## INTRODUCTION

In Fall 1992,the second author gave a course called "intermediate P.D.E" at the Courant Institute.The purpose of that course was to present some basic methods for obtaining various A Priori estimates for the second order partial differential equations of elliptic type with the particular emphasis on the maximal principles,Harnack inequalities and their applications.The equations one deals with are always linear though,obviously,they apply also to nonlinear problems. For students with some knowledge of real variables and Sobolev functions,they should be able to follow the course without much difficulties. The lecture notes was then taken by the first author.In 1995 at the university of Notre-Dame,the first author gave a similar course.The original notes was then much completed,and it resulted in the present form. We have no intention to give a complete account of the related theory.Our goal is simply that the notes may serve as bridge between elementary book of F.John [J] which studies equations of other type too,and somewhat advanced book of D.Gilbarg and N.Trudinger [GT] which gives relatively complete account of the theory of elliptic equations of second order.We also hope our notes can serve as a bridge between the recent elementary book of N.Krylov [K] on classical theory of elliptic equations developed before or around 1960's,and the book by Caffarelli and Xivier [CX] which studies fully nonlinear elliptic equations,the theory obtained in 1980's.

# CHAPTER 1

# HARMONIC FUNCTIONS

## GUIDE

The first chapter is rather elementary ,but it contains several important ideas of the whole subject.Thus it should be covered throughly.While doing the sections 1.1-1.2,the classical book of T.Rado[R] on subharmonic functions may be a very good reference.Also when one reads section 1.3,some statements concerning Hopf maximal principle in section 2.1 can be selected as exercises.The interior gradient estimates of section 2.3 follows from the same arguments as in the proof of the Proposition 3.2 of section 1.3.

In this chapter we will use various methods to study harmonic functions. These include mean value properties, fundamental solutions, maximum principles and energy method. Four sections in this chapter are relatively independent of each other.

## §1. Mean Value Property

We begin this section with the definition of mean value properties. We assume that $\Omega$ is a connected domain in $\mathbb{R}^n$.

*Definition.* For $u \in C(\Omega)$ we define (i) $u$ satisfies the first mean value property if

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(x)} u(y) dS_y \quad \text{for any } B_r(x) \subset \Omega;$$

(ii) $u$ satisfies the second mean value property if

$$u(x) = \frac{n}{\omega_n r^n} \int\limits_{B_r(x)} u(y) dy \quad \text{for any } B_r(x) \subset \Omega$$

where $\omega_n$ denotes the surface area of the unit sphere in $\mathbb{R}^n$.

*Remark.* These two definitions are equivalent. In fact if we write (i) as

$$u(x) r^{n-1} = \frac{1}{\omega_n} \int\limits_{\partial B_r(x)} u(y) dS_y,$$

we may integrate to get (ii). If we write (ii) as

$$u(x) r^n = \frac{n}{\omega_n} \int\limits_{B_r(x)} u(y) dy,$$

we may differentiate to get (i).

*Remark.* We may write the mean value properties in the following equivalent ways:
(i) $u$ satisfies the first mean value property if

$$u(x) = \frac{1}{\omega_n} \int\limits_{|w|=1} u(x + rw) dS_w \quad \text{for any } B_r(x) \subset \Omega;$$

(ii) $u$ satisfies the second mean value property if

$$u(x) = \frac{n}{\omega_n} \int_{|z| \leq 1} u(x + rz)\, dz \quad \text{for any } B_r(x) \subset \Omega.$$

Now we prove the maximum principle for the functions satisfying mean value properties.

**Proposition 1.1.** *If $u \in C(\bar{\Omega})$ satisfies the mean value property in $\Omega$, then $u$ assumes its maximum and minimum only on $\partial\Omega$ unless $u$ is constant.*

*Proof.* We only prove for the maximum. Set $\Sigma = \left\{ x \in \Omega;\ u(x) = M \equiv \max_{\bar{\Omega}} u \right\} \subset \Omega$. It is obvious that $\Sigma$ is relatively closed. Next we show that $\Sigma$ is open. For any $x_0 \in \Sigma$, take $\bar{B}_r(x_0) \subset \Omega$ for some $r > 0$. By the mean value property we have

$$M = u(x_0) = \frac{n}{\omega_n r^n} \int\limits_{B_r(x_0)} u(y)dy \leq M \frac{n}{\omega_n r^n} \int\limits_{B_r(x_0)} dy = M.$$

This implies $u = M$ in $B_r(x_0)$. Hence $\Sigma$ is both closed and open in $\Omega$. Therefore either $\Sigma = \phi$ or $\Sigma = \Omega$.

*Definition.* A function $u \in C^2(\Omega)$ is harmonic if $\triangle u = 0$ in $\Omega$.

**Theorem 1.2.** *Let $u \in C^2(\Omega)$ be harmonic in $\Omega$. Then $u$ satisfies the mean value property in $\Omega$.*

*Proof.* Take any ball $B_r(x) \subset \Omega$. For $\rho \in (0, r)$, we apply the divergence theorem in $B_\rho(x)$ and get

$$(*) \qquad \begin{aligned} \int_{B_\rho(x)} \triangle u(y)dy &= \int_{\partial B_\rho} \frac{\partial u}{\partial \nu}\, dS = \rho^{n-1} \int_{|w|=1} \frac{\partial u}{\partial \rho}(x + \rho w)dS_w \\ &= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|w|=1} u(x + \rho w)dS_w. \end{aligned}$$

Hence for harmonic function $u$ we have for any $\rho \in (0, r)$

$$\frac{\partial}{\partial \rho} \int_{|w|=1} u(x + \rho w)\ dS_w = 0.$$

Integrating from 0 to $r$ we obtain

$$\int_{|w|=1} u(x + rw)\ dS_w = \int_{|w|=1} u(x)\ dS_w = u(x)\omega_n$$

or

$$u(x) = \frac{1}{\omega_n} \int_{|w|=1} u(x + rw)\ dS_w = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y)dS_y.$$

*Remark.* For a function $u$ satisfying mean value property, $u$ is not required to be smooth. However a harmonic function is required to be $C^2$. We prove these two are equivalent.

**Theorem 1.3.** *If $u \in C(\Omega)$ has mean value property in $\Omega$, then $u$ is smooth and harmonic in $\Omega$.*

*Proof.* Choose $\varphi \in C_0^\infty(B_1(0))$ with $\int_{B_1(0)} \varphi = 1$ and $\varphi(x) = \psi(|x|)$, i.e.

$$\omega_n \int_0^1 r^{n-1}\psi(r)\, dr = 1.$$

We define $\varphi_\varepsilon(z) = \frac{1}{\varepsilon^n}\varphi\left(\frac{z}{\varepsilon}\right)$ for $\varepsilon > 0$. Now for any $x \in \Omega$ consider $\varepsilon < \text{dist }(x, \partial\Omega)$. Then we have

$$\begin{aligned}\int_\Omega u(y)\varphi_\varepsilon(y-x)dy &= \int u(x+y)\varphi_\varepsilon(y)dy = \frac{1}{\varepsilon^n}\int_{|y|<\varepsilon} u(x+y)\varphi\left(\frac{y}{\varepsilon}\right)dy \\ &= \int_{|y|<1} u(x+\varepsilon y)\varphi(y)dy \\ &= \int_0^1 r^{n-1}dr \int_{\partial B_1(0)} u(x+\varepsilon rw)\varphi(rw)dS_w \\ &= \int_0^1 \psi(r) r^{n-1}dr \int_{|w|=1} u(x+\varepsilon rw)dS_w \\ &= u(x)\omega_n \int_0^1 \psi(r)r^{n-1}dr = u(x)\end{aligned}$$

where in the last equality we used the mean value property. Hence we get

$$u(x) = (\varphi_\varepsilon * u)(x) \quad \text{for any } x \in \Omega_\varepsilon = \{y \in \Omega;\ d(y, \partial\Omega) > \varepsilon\}.$$

Therefore $u$ is smooth. Moreover, by formula $(*)$ in the proof of Theorem 1.2 and the mean value property we have

$$\int_{B_r(x)} \Delta u = r^{n-1}\frac{\partial}{\partial r}\int_{|w|=1} u(x+rw)dS_w = r^{n-1}\frac{\partial}{\partial r}\big(\omega_n u(x)\big) = 0 \text{ for any } B_r(x) \subset \Omega.$$

This implies $\Delta u = 0$ in $\Omega$.

*Remark.* By combining Theorems 1.1-1.3, we conclude that harmonic functions are smooth and satisfy the mean value property. Hence harmonic functions satisfy the maximum principle, a consequence of which is the uniqueness of solution to the following Dirichlet problem in a bounded domain

$$\begin{aligned}\Delta u &= f \text{ in } \Omega \\ u &= \varphi \text{ on } \partial\Omega\end{aligned}$$

for $f \in C(\Omega)$ and $\varphi \in C(\partial\Omega)$. In general uniqueness does not hold for unbounded domain. Consider the following Dirichlet problem in the unbounded domain $\Omega$

$$\Delta u = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

First consider the case $\Omega = \{x \in \mathbb{R}^n; |x| > 1\}$. For $n = 2, u(x) = \log |x|$ is a solution. Note $u \to \infty$ as $r \to \infty$. For $n \geq 3$, $u(x) = |x|^{2-n} - 1$ is a solution. Note $u \to -1$ as $r \to \infty$. Hence $u$ is bounded. Next, consider the upper half space $\Omega = \{x \in \mathbb{R}^n;\ x_n > 0\}$. Then $u(x) = x_n$ is a nontrivial solution, which is unbounded.

In the following we discuss the gradient estimates.

**Lemma 1.4.** *Suppose $u \in C(\bar{B}_R)$ is harmonic in $B_R = B_R(x_0)$. Then there holds*

$$|Du(x_0)| \leq \frac{n}{R} \max_{\bar{B}_R} |u|.$$

*Proof.* For simplicity we assume $u \in C^1(\bar{B}_R)$. Since $u$ is smooth, then $\Delta(D_{x_i} u) = 0$, i.e., $D_{x_i} u$ is also harmonic in $B_R$. Hence $D_{x_i} u$ satisfies the mean value property. By the divergence theorem we have

$$D_{x_i} u(x_0) = \frac{n}{\omega_n R^n} \int_{B_R(x_0)} D_{x_i} u(y) dy = \frac{n}{\omega_n R^n} \int_{\partial B_R(x_0)} u(y)\ \nu_i dS_y$$

which implies

$$|D_{x_i} u(x_0)| \leq \frac{n}{\omega_n R^n} \max_{\partial B_R} |u| \cdot \omega_n R^{n-1} \leq \frac{n}{R} \max_{\bar{B}_R} |u|.$$

**Lemma 1.5.** *Suppose $u \in C(\bar{B}_R)$ is a nonnegative harmonic function in $B_R = B_R(x_0)$. Then there holds*

$$|Du(x_0)| \leq \frac{n}{R} u(x_0).$$

*Proof.* As before by the divergence theorem and the nonnegativeness of $u$ we have

$$|D_{x_i} u(x_0)| \leq \frac{n}{\omega_n R^n} \int_{\partial B_R(x_0)} u(y)\ dS_y = \frac{n}{R} u(x_0)$$

where in the last equality we used the mean value property.

**Corollary 1.6.** *A harmonic function in $\mathbb{R}^n$ bounded from above or below is constant.*

*Proof.* Suppose $u$ is a harmonic function in $\mathbb{R}^n$. We will prove that $u$ is a constant if $u \geq 0$. In fact for any $x \in \mathbb{R}^n$ we apply Proposition 1.5 to $u$ in $B_R(x)$ and then let $R \to \infty$. We conclude that $Du(x) = 0$ for any $x \in \mathbb{R}^n$.

**Proposition 1.7.** *Suppose $u \in C(\bar{B}_R)$ is harmonic in $B_R = B_R(x_0)$. Then there holds for any multi-index $\alpha$ with $|\alpha| = m$*

$$|D^\alpha u(x_0)| \leq \frac{n^m e^{m-1} m!}{R^m} \max_{\bar{B}_R} |u|.$$

*Proof.* We prove by induction. It is true for $m = 1$ by Lemma 1.4. Assume it holds for $m$. Consider $m + 1$. For $0 < \theta < 1$, define $r = (1 - \theta)R \in (0, R)$. We apply Lemma 1.4 to $u$ in $B_r$ and get

$$|D^{m+1} u(x_0)| \leq \frac{n}{r} \max_{\bar{B}_r} |D^m u|.$$

By the induction assumption we have

$$\max_{\bar{B}_r} |D^m u| \leq \frac{n^m \cdot e^{m-1} \cdot m!}{(R - r)^m} \max_{\bar{B}_R} |u|.$$

Hence we obtain

$$|D^{m+1} u(x_0)| \leq \frac{n}{r} \cdot \frac{n^m e^{m-1} m!}{(R - r)^m} \max_{B_R} |u| = \frac{n^{m+1} e^{m-1} m!}{R^{m+1} \theta^m (1 - \theta)} \max_{\bar{B}_R} |u|.$$

Take $\theta = \frac{m}{m+1}$. This implies

$$\frac{1}{\theta^m (1 - \theta)} = \left(1 + \frac{1}{m}\right)^m (m + 1) < e(m + 1).$$

Hence the result is established for any single derivative. For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ we have

$$\alpha_1! \cdots \alpha_n! \leq (|\alpha|)!.$$

**Theorem 1.8.** *Harmonic function is analytic.*

*Proof.* Suppose $u$ is a harmonic function in $\Omega$. For fixed $x \in \Omega$, take $B_{2R}(x) \subset \Omega$ and $h \in \mathbb{R}^n$ with $|h| \leq R$. We have by Taylor expansion

$$u(x + h) = u(x) + \sum_{i=1}^{m-1} \frac{1}{i!} \left[ \left( h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n} \right)^i u \right] (x) + R_m(h)$$

where

$$R_m(h) = \frac{1}{m!}\left[\left(h_1\frac{\partial}{\partial x_1} + \cdots + h_n\frac{\partial}{\partial x_n}\right)^m u\right](x_1+\theta h_1, \ldots, x_n+\theta h_n)$$

for some $\theta \in (0,1)$. Note $x+h \in B_R(x)$ for $|h| < R$. Hence by Proposition 1.7 we obtain

$$|R_m(h)| \leq \frac{1}{m!}|h|^m \cdot n^m \cdot \frac{n^m e^{m-1} m!}{R^m} \max_{\bar{B}_{2R}}|u| \leq \left(\frac{|h|n^2 e}{R}\right)^m \max_{\bar{B}_{2R}}|u|.$$

Then for any $h$ with $|h|n^2 e < R/2$ there holds $R_m(h) \to 0$ as $m \to \infty$.

Next we prove the Harnack inequality.

**Theorem 1.9.** *Suppose $u$ is harmonic in $\Omega$. Then for any compact subset $K$ of $\Omega$ there exists a positive constant $C = (\Omega, K)$ such that if $u \geq 0$ in $\Omega$, then*

$$\frac{1}{C}u(y) \leq u(x) \leq Cu(y) \quad \textit{for any } x, y \in K.$$

*Proof.* By mean value property, we can prove if $B_{4R}(x_0) \subset \Omega$, then

$$\frac{1}{c}u(y) \leq u(x) \leq cu(y) \quad \text{for any } x, y \in B_R(x_0)$$

where $c$ is a positive constant depending only on $n$.

Now for the given compact subset $K$, take $x_1, \ldots, x_N \in K$ such that $\{B_R(x_i)\}$ covers $K$ with $4R < \text{dist}\,(K, \partial\Omega)$. Then we can choose $C = c^N$.

We finish this section by proving a result, originally due to Weyl. Suppose $u$ is harmonic in $\Omega$. Then we have by integrating by parts

$$\int_\Omega u\Delta\varphi = 0 \quad \text{for any } \varphi \in C_0^2(\Omega).$$

The converse is also true.

**Theorem 1.10.** *Suppose $u \in C(\Omega)$ satisfies*

$$\int_\Omega u\Delta\varphi = 0 \quad \textit{for any } \varphi \in C_0^2(\Omega). \tag{1}$$

*Then $u$ is harmonic in $\Omega$.*

*Proof.* We claim for any $B_r(x) \subset \Omega$ there holds

$$r \int_{\partial B_r(x)} u(y) dS_y = n \int_{B_r(x)} u(y) dy. \tag{2}$$

Then we have

$$\begin{aligned} & \frac{d}{dr}\left(\frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y) dS_y\right) \\ =& \frac{n}{\omega_n} \frac{d}{dr}\left(\frac{1}{r^n} \int_{B_r(x)} u(y) dy\right) \\ =& \frac{n}{\omega_n}\left\{-\frac{n}{r^{n+1}} \int_{B_r(x)} u(y) dy + \frac{1}{r^n} \int_{\partial B_r(x)} u(y) dS_y\right\} = 0. \end{aligned}$$

This implies

$$\frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y) dS_y = \text{const}.$$

This constant is $u(x)$ if we let $r \to 0$. Hence we have

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y) dS_y \text{ for any } B_r(x) \subset \Omega.$$

Next we prove (2) for $n \geq 3$. For simplicity we assume that $x = 0$. Set

$$\varphi(y, r) = \begin{cases} (|y|^2 - r^2)^n & |y| \leq r \\ 0 & |y| > r \end{cases}$$

and then $\varphi_k(y, r) = (|y|^2 - r^2)^{n-k}\big(2(n-k+1)|y|^2 + n(|y|^2 - r^2)\big)$ for $|y| \leq r$ and $k = 2, 3, \ldots, n$. Direct calculation shows $\varphi(\cdot, r) \in C_0^2(\Omega)$ and

$$\Delta_y \varphi(y, r) = \begin{cases} 2n\varphi_2(y, r) & |y| \leq r \\ 0 & |y| > r \end{cases}.$$

By assumption (1) we have

$$\int_{B_r(0)} u(y) \varphi_2(y, r) dy = 0.$$

Now we prove if for some $k = 2, \cdots, n-1$,

$$\int_{B_r(0)} u(y) \varphi_k(y, r) dy = 0 \tag{3}$$

then

$$\int_{B_r(0)} u(y)\varphi_{k+1}(y,r)dy = 0. \tag{4}$$

In fact we differentiate (3) with respect to $r$ and get

$$\int_{\partial B_r(0)} u(y)\varphi_k(y,r)dy + \int_{B_r(0)} u(y)\frac{\partial \varphi_k}{\partial r}(y,r)dy = 0.$$

For $2 \le k < n$, $\varphi_k(y,r) = 0$ for $|y| = r$. Then we have

$$\int_{B_r(0)} u(y)\frac{\partial \varphi_k}{\partial r}(y,r)dy = 0.$$

Direct calculation yields $\frac{\partial \varphi_k}{\partial r}(y,r) = (-2r)(n-k+1)\varphi_{k+1}(y,r)$. Hence we have (4). Therefore by taking $k = n-1$ in (4) we conclude

$$\int_{B_r(0)} u(y)\big((n+2)|y|^2 - nr^2\big)dy = 0.$$

Differentiating with respect to $r$ again we get (2).

## §2. Fundamental Solutions

We begin this section by seeking a harmonic function $u$, i.e., $\Delta u = 0$, in $\mathbb{R}^n$ which depends only on $r = |x-a|$ for some fixed $a \in \mathbb{R}^n$. We set $v(r) = u(x)$. This implies

$$v'' + \frac{n-1}{r}v' = 0$$

and hence

$$v(r) = \begin{cases} c_1 + c_2 \log r, & n = 2 \\ c_3 + c_4 r^{2-n}, & n \ge 3 \end{cases}$$

where $c_i$ are constants for $i = 1,2,3,4$. We are interested in a function with singularity such that

$$\int_{\partial B_r} \frac{\partial u}{\partial r} dS = 1 \quad \text{for any } r > 0.$$

Hence we set for any fixed $a \in \mathbb{R}^n$

$$\Gamma(a,x) = \frac{1}{2\pi}\log|a-x| \quad \text{for } n = 2$$

$$\Gamma(a,x) = \frac{1}{\omega_n(2-n)}|a-x|^{2-n} \quad \text{for } n \ge 3.$$

To summarize we have that for fixed $a \in \mathbb{R}^n$, $\Gamma(a,x)$ is harmonic at $x \neq a$, i.e.,

$$\Delta_x \Gamma(a,x) = 0 \quad \text{for any } x \neq a$$

and has a singularity at $x = a$. Moreover it satisfies

$$\int_{\partial B_r(a)} \frac{\partial \Gamma}{\partial n_x}(a,x)\, dS_x = 1 \qquad \text{for any } r > 0.$$

Now we prove the Green's identity.

**Theorem 2.1.** *Suppose $\Omega$ is a bounded domain in $\mathbb{R}^n$ and that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. Then for any $a \in \Omega$ there holds*

$$u(a) = \int_{\Omega} \Gamma(a,x)\Delta u(x)dx - \int_{\partial\Omega} \left( \Gamma(a,x)\frac{\partial u}{\partial n_x}(x) - u(x)\frac{\partial \Gamma}{\partial n_x}(a,x) \right) dS_x.$$

*Remark.* (i) For any $a \in \Omega$, $\Gamma(a,\cdot)$ is integrable in $\Omega$ although it has a singularity.

(ii) For $a \notin \bar{\Omega}$, the expression in the right side gives zero.

(iii) By letting $u = 1$ we have $\int_{\partial\Omega} \frac{\partial\Gamma}{\partial n_x}(a,x)dS_x = 1$ for any $a \in \Omega$.

*Proof.* We apply Green's formula to $u$ and $\Gamma(a,\cdot)$ in the domain $\Omega \setminus B_r(a)$ for small $r > 0$ and get

$$\int_{\Omega\setminus B_r(a)} (\Gamma\Delta u - u\Delta\Gamma)dx = \int_{\partial\Omega} \left( \Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n} \right) dS_x - \int_{\partial B_r(a)} \left( \Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n} \right) dS_x.$$

Note $\Delta\Gamma = 0$ in $\Omega \setminus B_r(a)$. Then we have

$$\int_{\Omega} \Gamma\Delta u dx = \int_{\partial\Omega} \left( \Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n} \right) dS_x - \lim_{r\to 0} \int_{\partial B_r(a)} \left( \Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n} \right) dS_x.$$

For $n \geq 3$, we get by definition of $\Gamma$

$$\begin{aligned} \left| \int_{\partial B_r(a)} \Gamma\frac{\partial u}{\partial n} dS \right| &= \left| \frac{1}{(2-n)\omega_n} r^{2-n} \int_{\partial B_r(a)} \frac{\partial u}{\partial n} dS \right| \\ &\leq \frac{r}{n-2} \sup_{\partial B_r(a)} |Du| \to 0 \text{ as } r \to 0 \end{aligned}$$

and

$$\int_{\partial B_r(a)} u\frac{\partial \Gamma}{\partial n}\,dS = \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(a)} udS \to u(a) \text{ as } \; r\to 0.$$

For $n = 2$, we get the same conclusion similarly.

*Remark.* We may employ the local version of the Green's identity to get gradient estimates without using mean value property. Suppose $u \in C(\bar{B}_1)$ is harmonic in $B_1$. For any fixed $0 < r < R < 1$ choose a cut-off function $\varphi \in C_0^\infty(B_R)$ such that $\varphi = 1$ in $B_r$ and $0 \le \varphi \le 1$. Apply the Green's formula to $u$ and $\varphi\Gamma(a,\cdot)$ in $B_1 \setminus B_\rho(a)$ for $a \in B_r$ and $\rho$ small enough. We proceed as in the proof of Theorem 2.1 and we obtain

$$u(a) = -\int_{r<|x|<R} u(x)\ \Delta_x\big(\varphi(x)\Gamma(a,x)\big)dx \qquad \text{for any } a \in B_r(0).$$

Hence one may prove (without using mean value property)

$$\sup_{B_{\frac{1}{2}}} |u| \le c\left(\int_{B_1} |u|^p\right)^{\frac{1}{p}}$$

and

$$\sup_{B_{\frac{1}{2}}} |Du| \le c\max_{B_1} |u|$$

where $c$ is a constant depending only on $n$.

Now we begin to discuss the Green's functions. Suppose $\Omega$ is bounded domain in $\mathbb{R}^n$. Let $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. We have by Theorem 2.1 for any $x \in \Omega$

$$u(x) = \int_\Omega \Gamma(x,y)\Delta u(y)dy - \int_{\partial\Omega}\left(\Gamma(x,y)\frac{\partial u}{\partial n_y}(y) - u(y)\frac{\partial \Gamma}{\partial n_y}(x,y)\right)dS_y.$$

If $u$ solves the following Dirichlet boundary value problem

$$(*) \qquad \begin{cases} \Delta u = f & \text{in } \; \Omega \\ u = \varphi & \text{on } \; \partial\Omega \end{cases}$$

for some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$, then $u$ can be expressed in terms of $f$ and $\varphi$, with one *unknown term*. We want to eliminate this term by adjusting $\Gamma$.

For any fixed $x \in \Omega$, consider

$$\gamma(x,y) = \Gamma(x,y) + \Phi(x,y)$$

for some $\Phi(x,\cdot)\in C^2(\bar\Omega)$ with $\Delta_y\Phi(x,y)=0$ in $\Omega$. Then Theorem 2.1 can be expressed as follows for any $x\in\Omega$

$$u(x)=\int_\Omega \gamma(x,y)\Delta u(y)dy-\int_{\partial\Omega}\left(\gamma(x,y)\frac{\partial u}{\partial n_y}(y)-u(y)\frac{\partial\gamma}{\partial n_y}(x,y)\right)dS_y$$

since the extra $\Phi(x,\cdot)$ is harmonic. Now by choosing $\Phi$ appropriately, we are led to the important concept of Green's function.

For each fixed $x\in\Omega$ choose $\Phi(x,\cdot)\in C^1(\bar\Omega)\cap C^2(\Omega)$ such that

$$\begin{cases}\Delta_y\Phi(x,y)=0 & \text{for } \ y\in\Omega\\ \Phi(x,y)=-\Gamma(x,y) & \text{for } \ y\in\partial\Omega.\end{cases}$$

Denote the resulting $\gamma(x,y)$ by $G(x,y)$, which is called Green's function. If such $G$ exists, then solution $u$ to the Dirichlet problem $(*)$ can be expressed as

$$u(x)=\int_\Omega G(x,y)f(y)dy+\int_{\partial\Omega}\varphi(y)\frac{\partial G}{\partial n_y}(x,y)dS_y.$$

Note that Green's function $G(x,y)$ is defined as a function of $y\in\bar\Omega$ for each fixed $x\in\Omega$. Now we discuss some properties of $G$ as function of $x$ and $y$. First observation is that the Green's function is unique. This is proved by the maximum principle since the difference of two Green's functions are harmonic in $\Omega$ with zero boundary value. In fact, we have more.

**Proposition 2.2.** *Green's function $G(x,y)$ is symmetric in $\Omega\times\Omega$, i.e., $G(x,y)=G(y,x)$ for $x\neq y\in\Omega$.*

*Proof.* Pick $x_1,x_2\in\Omega$ with $x_1\neq x_2$. Choose $r>0$ small such that $B_r(x_1)\cap B_r(x_2)=\phi$. Set $G_1(y)=G(x_1,y)$ and $G_2(y)=G(x_2,y)$ . We apply Green's formula in $\Omega\setminus B_r(x_1)\cup B_r(x_2)$ and get

$$\begin{aligned}\int_{\Omega\setminus B_r(x_1)\cup B_r(x_2)}(G_1\Delta G_2-G_2\Delta G_1)&=\int_{\partial\Omega}\left(G_1\frac{\partial G_2}{\partial n}-G_2\frac{\partial G_1}{\partial n}\right)dS\\ -\int_{\partial B_r(x_1)}\left(G_1\frac{\partial G_2}{\partial n}-G_2\frac{\partial G_1}{\partial n}\right)dS&-\int_{\partial B_r(x_2)}\left(G_1\frac{\partial G_2}{\partial n}-G_2\frac{\partial G_1}{\partial n}\right)dS.\end{aligned}$$

Since $G_i$ is harmonic for $y\neq x_i$, $i=1,2$, and vanishes on $\partial\Omega$ we have

$$\int_{\partial B_r(x_1)}\left(G_1\frac{\partial G_2}{\partial n}-G_2\frac{\partial G_1}{\partial n}\right)dS+\int_{\partial B_r(x_2)}\left(G_1\frac{\partial G_2}{\partial n}-G_2\frac{\partial G_1}{\partial n}\right)dS=0.$$

Note the left side has the same limit as the left side in the following as $r \to 0$

$$\int_{\partial B_r(x_1)} \left( \Gamma \frac{\partial G_2}{\partial n} - G_2 \frac{\partial \Gamma}{\partial n} \right) dS + \int_{\partial B_r(x_2)} \left( G_1 \frac{\partial \Gamma}{\partial n} - \Gamma \frac{\partial G_1}{\partial n} \right) dS = 0.$$

While we have

$$\int_{\partial B_r(x_1)} \Gamma \frac{\partial G_2}{\partial n} dS \to 0, \quad \int_{\partial B_r(x_2)} \Gamma \frac{\partial G_1}{\partial n} dS \to 0 \quad \text{as} \ \ r \to 0$$

and

$$\int_{\partial B_r(x_1)} G_2 \frac{\partial \Gamma}{\partial n} dS \to G_2(x_1), \quad \int_{\partial B_r(x_2)} G_1 \frac{\partial \Gamma}{\partial n} dS \to G_1(x_2) \quad \text{as} \ \ r \to 0.$$

This implies $G_2(x_1) - G_1(x_2) = 0$, or $G(x_2, x_1) = G(x_1, x_2)$.

**Proposition 2.3.** *There holds for $x, y \in \Omega$ with $x \neq y$*

$$0 > G(x,y) > \Gamma(x,y) \quad \textit{for } n \geq 3$$
$$0 > G(x,y) > \Gamma(x,y) - \frac{1}{2\pi} \log \textit{diam}(\Omega) \quad \textit{for } n = 2.$$

*Proof.* Fix $x \in \Omega$ and write $G(y) = G(x,y)$. Since $\lim_{y \to x} G(y) = -\infty$ then there exists an $r > 0$ such that $G(y) < 0$ in $B_r(x)$. Note that $G$ is harmonic in $\Omega \setminus B_r(x)$ with $G = 0$ on $\partial\Omega$ and $G < 0$ on $\partial B_r(x)$. Maximum principle implies $G(y) < 0$ in $\Omega \setminus B_r(x)$ for such $r > 0$. Next, consider the other part of the inequality. Recall the definition of the Green's function

$$G(x,y) = \Gamma(x,y) + \Phi(x,y)$$

where

$$\Delta \Phi = 0 \quad \text{in} \ \ \Omega$$
$$\Phi = -\Gamma \quad \text{on} \ \ \partial\Omega.$$

For $n \geq 3$, we have

$$\Gamma(x,y) = \frac{1}{(2-n)\omega_n} |x-y|^{2-n} < 0 \quad \text{for } y \in \partial\Omega$$

which implies $\Phi(x, \cdot) > 0$ on $\partial\Omega$. By the maximum principle, we have $\Phi > 0$ in $\Omega$. For $n = 2$ we have

$$\Gamma(x,y) = \frac{1}{2\pi} \log|x-y| \leq \frac{1}{2\pi} \log \text{diam}(\Omega) \quad \text{for } y \in \partial\Omega.$$

Hence the maximum principle implies $\Phi > -\frac{1}{2\pi} \log \text{diam}(\Omega)$ in $\Omega$.

We may calculate Green's functions for some special domains.

**Proposition 2.4.** *The Green's function for the ball $B_R(0)$ is given by*

*(i) for $n \geq 3$*

$$G(x,y) = \frac{1}{(2-n)\omega_n}\left(|x-y|^{2-n} - \left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|^{2-n}\right);$$

*(ii) for $n = 2$*

$$G(x,y) = \frac{1}{2\pi}\left(\log|x-y| - \log\left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|\right).$$

*Proof.* Fix $x \neq 0$ with $|x| < R$. Consider $X \in \mathbb{R}^n \setminus \bar{B}_R$ with $X$ the multiple of $x$ and $|X| \cdot |x| = R^2$, i.e., $X = \frac{R^2}{|x|^2}x$. In other words $X$ and $x$ are reflexive of each other with respect to the sphere $\partial B_R$. Note the map $x \longmapsto X$ is conformal, i.e., preserves angles. If $|y| = R$, we have by similarity of triangles

$$\frac{|x|}{R} = \frac{R}{|X|} = \frac{|y-x|}{|y-X|}$$

which implies

$$|y-x| = \frac{|x|}{R}|y-X| = \left|\frac{|x|}{R}y - \frac{R}{|x|}x\right| \quad \text{for any } y \in \partial B_R. \tag{1}$$

Therefore, in order to have zero boundary value, we take for $n \geq 3$

$$G(x,y) = \frac{1}{(2-n)\omega_n}\left(\frac{1}{|x-y|^{n-2}} - \left(\frac{R}{|x|}\right)^{n-2}\frac{1}{|y-X|^{n-2}}\right).$$

The case $n = 2$ is similar.

Next, we calculate the normal derivative of Green's function on the sphere.

**Corollary 2.5.** *Suppose $G$ is the Green's function in $B_R(0)$. Then there holds*

$$\frac{\partial G}{\partial n}(x,y) = \frac{R^2 - |x|^2}{\omega_n R|x-y|^n} \quad \textit{for any } x \in B_R \textit{ and } y \in \partial B_R.$$

*Proof.* We just consider the case $n \geq 3$. Recall with $X = R^2x/|x|^2$

$$G(x,y) = \frac{1}{(2-n)\omega_n}\left(|x-y|^{2-n} - \left(\frac{R}{|x|}\right)^{n-2}|y-X|^{2-n}\right) \quad \text{for } x \in B_R, y \in \partial B_R.$$

Hence we have for such $x$ and $y$

$$D_{y_i}G(x,y) = -\frac{1}{\omega_n}\left(\frac{x_i - y_i}{|x-y|^n} - \left(\frac{R}{|x|}\right)^{n-2} \cdot \frac{X_i - y_i}{|X-y|^n}\right) = \frac{y_i}{\omega_n R^2}\frac{R^2 - |x|^2}{|x-y|^n}$$

by (1) in the proof of Proposition 2.4. We obtain with $n_i = \frac{y_i}{R}$ for $|y| = R$

$$\frac{\partial G}{\partial n}(x,y) = \sum_{i=1}^{n} n_i D_{y_i}G(x,y) = \frac{1}{w_n R} \cdot \frac{R^2 - |x|^2}{|x-y|^n}.$$

Denote by $K(x,y)$ the function in Corollary 2.5 for $x \in \Omega, y \in \partial\Omega$. It is called Poisson kernel and has the following properties:

(i) $K(x,y)$ is smooth for $x \neq y$.

(ii) $K(x,y) > 0$ for $|x| < R$

(iii) $\int_{|y|=R} K(x,y)dS_y = 1$ for any $|x| < R$.

The following result gives the existence of harmonic functions in balls with prescribed Dirichlet boundary value.

**Theorem 2.6 (Poisson Integral Formula).** *For $\varphi \in C(\partial B_R(0))$, the function $u$ defined by*

$$u(x) = \begin{cases} \int_{\partial B_R(0)} K(x,y)\varphi(y)dS_y & |x| < R \\ \varphi(x) & |x| = R \end{cases}$$

*satisfies $u \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ and*

$$\begin{cases} \Delta u = 0 & in\ \Omega \\ u = \varphi & on\ \partial\Omega. \end{cases}$$

For the proof, see F. John P107 - P108.

*Remark.* In Poisson integral formula, by letting $x = 0$, we have

$$u(0) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(0)} \varphi(y)dS_y$$

which is the mean value property.

**Lemma 2.7 (Harnack's Inequality).** *Suppose $u$ is harmonic in $B_R(x_0)$ and $u \geq 0$. Then there holds*

$$\left(\frac{R}{R+r}\right)^{n-2}\frac{R-r}{R+r}u(x_0) \leq u(x) \leq \left(\frac{R}{R-r}\right)^{n-2}\frac{R+r}{R-r}u(x_0)$$

*where $r = |x - x_0| < R$.*

*Proof.* We may assume $x_0 = 0$ and $u \in C(\bar{B}_R)$. Note that $u$ is given by Poisson Integral Formula

$$u(x) = \frac{1}{\omega_n R}\int_{\partial B_R}\frac{R^2 - |x|^2}{|y-x|^n}u(y)dS_y.$$

Since $R - |x| \leq |y - x| \leq R + |x|$ for $|y| = R$, we have

$$\frac{1}{\omega_n R}\cdot\frac{R-|x|}{R+|x|}\left(\frac{1}{R+|x|}\right)^{n-2}\int_{\partial B_R}u(y)dS_y \leq u(x)$$
$$\leq \frac{1}{\omega_n R}\cdot\frac{R+|x|}{R-|x|}\left(\frac{1}{R-|x|}\right)^{n-2}\int_{\partial B_R}u(y)dS_y.$$

Mean value property implies

$$u(0) = \frac{1}{\omega_n R^{n-1}}\int_{\partial B_R}u(y)dS_y.$$

This finishes the proof.

**Corollary 2.8.** *If harmonic function $u$ in $\mathbb{R}^n$ is bounded above or below, then $u \equiv$ const.*

*Proof.* We assume $u \geq 0$ in $\mathbb{R}^n$. Take any point $x \in \mathbb{R}^n$ and apply Lemma 2.7 to any ball $B_R(0)$ with $R > |x|$. We obtain

$$\left(\frac{R}{R+|x|}\right)^{n-2}\frac{R-|x|}{R+|x|}u(0) \leq u(x) \leq \left(\frac{R}{R-|x|}\right)^{n-2}\frac{R+|x|}{R-|x|}u(0)$$

which implies $u(x) = u(0)$ by letting $R \to +\infty$.

Next we prove a result concerning the removable singularity.

**Theorem 2.9.** *Suppose $u$ is harmonic in $B_R \setminus \{0\}$ and satisfies*

$$u(x) = \begin{cases} o(\log|x|), & n = 2 \\ o(|x|^{2-n}), & n \geq 3 \end{cases} \quad \text{as } |x| \to 0.$$

*Then $u$ can be defined at 0 so that it is $C^2$ and harmonic in $B_R$.*

*Proof.* Assume $u$ is continuous in $0 < |x| \le R$. Let $v$ solve

$$\begin{cases} \Delta v = 0 \text{ in } B_R \\ v = u \text{ on } \partial B_R. \end{cases}$$

We will prove $u = v$ in $B_R \setminus \{0\}$. Set $w = v - u$ in $B_R \setminus \{0\}$ and $M_r = \max\limits_{\partial B_r}|w|$. We prove for $n \ge 3$. It is obvious that

$$|w(x)| \le M_r \cdot \frac{r^{n-2}}{|x|^{n-2}} \text{ on } \partial B_r.$$

Note $w$ and $\frac{1}{|x|^{n-2}}$ are harmonic in $B_R \setminus B_r$. Hence maximum principle implies

$$|w(x)| \le M_r \cdot \frac{r^{n-2}}{|x|^{n-2}} \text{ for any } x \in B_R \setminus B_r$$

where $M_r = \max\limits_{\partial B_r}|v - u| \le \max\limits_{\partial B_r}|v| + \max\limits_{\partial B_r}|u| \le M + \max\limits_{\partial B_r}|u|$ with $M = \max\limits_{\partial B_R}|u|$. Hence we have for each fixed $x \ne 0$

$$|w(x)| \le \frac{r^{n-2}}{|x|^{n-2}}M + \frac{1}{|x|^{n-2}} r^{n-2} \max_{\partial B_r}|u| \to 0 \text{ as } r \to 0,$$

that is $w = 0$ in $B_R \setminus \{0\}$.

## §3. Maximum Principles

In this section we will use the maximum principle to derive the interior gradient estimate and the Harnack inequality.

**Theorem 3.1.** *Suppose $u \in C^2(B_1) \cap C(\bar{B}_1)$ is a subharmonic function in $B_1$, i.e., $\triangle u \ge 0$. Then there holds*

$$\sup_{B_1} u \le \sup_{\partial B_1} u.$$

*Proof.* For $\varepsilon > 0$ we consider $u_\varepsilon(x) = u(x) + \varepsilon|x|^2$ in $B_1$. Then simple calculation yields

$$\triangle u_\varepsilon = \triangle u + 2n\varepsilon \ge 2n\varepsilon > 0.$$

It is easy to see, by contradiction argument, that $u_\varepsilon$ can not have an interior maximum, in particular,

$$\sup_{B_1} u_\varepsilon \leq \sup_{\partial B_1} u_\varepsilon.$$

Therefore we have

$$\sup_{B_1} u \leq \sup_{B_1} u_\varepsilon \leq \sup_{\partial B_1} u + \varepsilon.$$

We finish the proof by letting $\varepsilon \to 0$.

*Remark.* The result still holds if $B_1$ is replaced by any bounded domain.

Next result is the interior gradient estimate for harmonic functions. The method is due to Bernstein back in 1910.

**Proposition 3.2.** *Suppose $u$ is harmonic in $B_1$. Then there holds*

$$\sup_{B_{\frac{1}{2}}} |Du| \leq c \sup_{\partial B_1} |u|$$

*where $c = c(n)$ is a positive constant. In particular for any $\alpha \in [0,1]$ there holds*

$$|u(x) - u(y)| \leq c|x-y|^\alpha \sup_{\partial B_1} |u| \quad \textit{for any } x, y \in B_{\frac{1}{2}}$$

*where $c = c(n, \alpha)$ is a positive constant.*

*Proof.* Direct calculation shows that

$$\triangle(|Du|^2) = 2\sum_{i,j=1}^n (D_{ij}u)^2 + 2\sum_{i=1}^n D_i u D_i(\triangle u) = 2\sum_{i,j=1}^n (D_{ij}u)^2$$

where we used $\triangle u = 0$ in $B_1$. Hence $|Du|^2$ is a subharmonic function. To get interior estimates we need a cut-off function. For any $\varphi \in C_0^1(B_1)$ we have

$$\triangle(\varphi|Du|^2) = (\triangle\varphi)|Du|^2 + 4\sum_{i,j=1}^n D_i\varphi D_j u D_{ij}u + 2\varphi\sum_{i,j=1}^n (D_{ij}u)^2.$$

By taking $\varphi = \eta^2$ for some $\eta \in C_0^1(B_1)$ with $\eta \equiv 1$ in $B_{1/2}$ we obtain by Hölder inequality

$$\begin{aligned}\triangle(\eta^2|Du|^2) &= 2\eta\triangle\eta|Du|^2 + 2|D\eta|^2|Du|^2 + 8\eta\sum_{i,j=1}^n D_i\eta D_j u D_{ij}u + 2\eta^2\sum_{i,j=1}^n (D_{ij}u)^2 \\ &\geq \left(2\eta\triangle\eta - 6|D\eta|^2\right)|Du|^2 \geq -C|Du|^2\end{aligned}$$

where $C$ is a positive constant depending only on $\eta$. Note $\triangle(u^2) = 2|Du|^2 + 2u\triangle u = 2|Du|^2$ since $u$ is harmonic. By taking $\alpha$ large enough we get

$$\triangle(\eta^2|Du|^2 + \alpha u^2) \geq 0.$$

We may apply Theorem 3.1 (the maximum principle) to get the result.

Next we derive the Harnack inequality.

**Lemma 3.3.** *Suppose $u$ is a nonnegative harmonic function in $B_1$. Then there holds*

$$\sup_{B_{\frac{1}{2}}} |D \log u| \leq C$$

*where $C = C(n)$ is a positive constant.*

*Proof.* We may assume $u > 0$ in $B_1$. Set $v = \log u$. Then direct calculation shows

$$\triangle v = -|Dv|^2.$$

We need interior gradient estimate on $v$. Set $w = |Dv|^2$. Then we get

$$\triangle w + 2\sum_{i=1}^{n} D_i v D_i w = 2\sum_{i,j=1}^{n}(D_{ij}v)^2.$$

As before we need a cut-off function. First note

$$\sum_{i,j=1}^{n}(D_{ij}v)^2 \geq \sum_{i}^{n}(D_{ii}v)^2 \geq \frac{1}{n}(\triangle v)^2 = \frac{|Dv|^4}{n} = \frac{w^2}{n}. \tag{1}$$

Take a nonnegative function $\varphi \in C_0^1(B_1)$. We obtain by Hölder inequality

$$\begin{aligned}
&\triangle(\varphi w) + 2\sum_{i=1}^{n} D_i v D_i(\varphi w)\\
=&2\varphi\sum_{i,j=1}^{n}(D_{ij}v)^2 + 4\sum_{i,j=1}^{n} D_i\varphi D_j v D_{ij}v + 2w\sum_{i=1}^{n} D_i\varphi D_i v + (\triangle\varphi)w\\
\geq&\varphi\sum_{i,j=1}^{n}(D_{ij}v)^2 - 2|D\varphi||Dv|^3 - \left(|\triangle\varphi| + C\frac{|D\varphi|^2}{\varphi}\right)|Dv|^2
\end{aligned}$$

if $\varphi$ is chosen such that $|D\varphi|^2/\varphi$ is bounded in $B_1$. Choose $\varphi = \eta^4$ for some $\eta \in C_0^1(B_1)$. Hence for such fixed $\eta$ we obtain by (1)

$$\begin{aligned}
&\triangle(\eta^4 w) + 2\sum_{i=1}^{n} D_i v D_i(\eta^4 w) \\
\geq &\frac{1}{n}\eta^4|Dv|^4 - C\eta^3|D\eta||Dv|^3 - 4\eta^2(\eta\triangle\eta + C|D\eta|^2)|Dv|^2 \\
\geq &\frac{1}{n}\eta^4|Dv|^4 - C\eta^3|Dv|^3 - C\eta^2|Dv|^2
\end{aligned}$$

where $C$ is a positive constant depending only on $n$ and $\eta$. Hence we get by Hölder inequality

$$\triangle(\eta^4 w) + 2\sum_{i=1}^{n} D_i v D_i(\eta^4 w) \geq \frac{1}{n}\eta^4 w^2 - C$$

where $C$ is a positive constant depending only on $n$ and $\eta$.

Suppose $\eta^4 w$ attains its maximum at $x_0 \in B_1$. Then $D(\eta^4 w) = 0$ and $\triangle(\eta^4 w) \leq 0$ at $x_0$. Hence there holds

$$\eta^4 w^2(x_0) \leq C(n, \eta).$$

If $w(x_0) \geq 1$, then $\eta^4 w(x_0) \leq C(n)$. Otherwise $\eta^4 w(x_0) \leq w(x_0) \leq \eta^4(x_0)$. In both cases we conclude

$$\eta^4 w \leq C(n, \eta) \quad \text{in} \quad B_1.$$

**Corollary 3.4.** *Suppose $u$ is a nonnegative harmonic function in $B_1$. Then there holds*

$$u(x_1) \leq Cu(x_2) \text{ for any } x_1, x_2 \in B_{\frac{1}{2}}$$

*where $C$ is a positive constant depending only on $n$.*

*Proof.* We may assume $u > 0$ in $B_1$. For any $x_1, x_2 \in B_{\frac{1}{2}}$ by simple integration we obtain with Lemma 3.3

$$\log\frac{u(x_1)}{u(x_2)} \leq |x_1 - x_2| \int_0^1 |D\log u(tx_2 + (1-t)x_1)|dt \leq C|x_1 - x_2|.$$

Next we prove a quantitative Hopf Lemma.

**Proposition 3.5.** *Suppose $u \in C(\bar{B}_1)$ is a harmonic function in $B_1 = B_1(0)$. If $u(x) < u(x_0)$ for any $x \in \bar{B}_1$ and some $x_0 \in \partial B_1$, then there holds*

$$\frac{\partial u}{\partial n}(x_0) \geq C\Big(u(x_0) - u(0)\Big)$$

*where $C$ is a positive constant depending only on $n$.*

*Proof.* Consider a positive function $v$ in $B_1$ defined by

$$v(x) = e^{-\alpha|x|^2} - e^{-\alpha}.$$

It is easy to see

$$\triangle v(x) = e^{-\alpha|x|^2}(-2\alpha n + 4\alpha^2|x|^2) > 0 \text{ for any } |x| \geq \frac{1}{2}$$

if $\alpha \geq 2n+1$. Hence for such fixed $\alpha$ the function $v$ is subharmonic in the region $A = B_1 \setminus B_{1/2}$. Now define for $\varepsilon > 0$

$$h_\varepsilon(x) = u(x) - u(x_0) + \varepsilon v(x).$$

This is also a subharmonic function, i.e., $\triangle h_\varepsilon \geq 0$ in $A$. Obviously $h_\varepsilon \leq 0$ on $\partial B_1$ and $h_\varepsilon(x_0) = 0$. Since $u(x) < u(x_0)$ for $|x| = 1/2$ we may take $\varepsilon > 0$ small such that $h_\varepsilon(x) < 0$ for $|x| = 1/2$. Therefore by Theorem 3.1 $h_\varepsilon$ assumes at the point $x_0$ its maximum in $A$. This implies

$$\frac{\partial h_\varepsilon}{\partial n}(x_0) \geq 0$$

or

$$\frac{\partial u}{\partial n}(x_0) \geq -\varepsilon\frac{\partial v}{\partial n}(x_0) = 2\alpha\varepsilon e^{-\alpha} > 0.$$

Note so far we only used the subharmonicity of $u$. We estimate $\varepsilon$ as follows. Set $w(x) = u(x_0) - u(x) > 0$ in $B_1$. Obviously $w$ is a harmonic function in $B_1$. By Corollary 3.4 (Harnack inequality) there holds

$$\inf_{B_{\frac{1}{2}}} w \geq c(n)w(0)$$

or

$$\max_{B_{\frac{1}{2}}} u \leq u(x_0) - c(n)\Big(u(x_0) - u(0)\Big).$$

Hence we may take

$$\varepsilon = \delta c(n)\Big(u(x_0) - u(0)\Big)$$

for $\delta$ small, depending on $n$. This finishes the proof.

To finish this section we prove a global Hölder continuity result.

**Lemma 3.6.** *Suppose $u \in C(\bar{B}_1)$ is a harmonic function in $B_1$ with $u = \varphi$ on $\partial B_1$. If $\varphi \in C^\alpha(\partial B_1)$ for some $\alpha \in (0,1)$, then $u \in C^{\alpha/2}(\bar{B}_1)$. Moreover there holds*

$$\|u\|_{C^{\frac{\alpha}{2}}(\bar{B}_1)} \leq C\|\varphi\|_{C^\alpha(\partial B_1)}$$

*where $C$ is a positive constant depending only on $n$ and $\alpha$.*

*Proof.* First the maximum principle implies that $\inf_{\partial B_1} \varphi \leq u \leq \sup_{\partial B_1} \varphi$ in $B_1$. Next we claim for any $x_0 \in \partial B_1$ there holds

$$\sup_{x \in B_1} \frac{|u(x) - u(x_0)|}{|x - x_0|^{\frac{\alpha}{2}}} \leq 2^{\frac{\alpha}{2}} \sup_{x \in \partial B_1} \frac{|\varphi(x) - \varphi(x_0)|}{|x - x_0|^\alpha}. \tag{1}$$

Lemma 3.6 follows easily from (1). For any $x, y \in B_1$, set $d_x = \text{dist}(x, \partial B_1)$ and $d_y = \text{dist}(y, \partial B_1)$. Suppose $d_y \leq d_x$. Take $x_0, y_0 \in \partial B_1$ such that $|x - x_0| = d_x$ and $|y - y_0| = d_y$. Assume first that $|x - y| \leq d_x/2$. Then $y \in \bar{B}_{d_x/2}(x) \subset B_{d_x}(x) \subset B_1$. We apply Theorem 3.2 (scaled version) to $u - u(x_0)$ in $B_{d_x}(x)$ and get by (1)

$$d_x^{\frac{\alpha}{2}} \frac{|u(x) - u(y)|}{|x - y|^{\frac{\alpha}{2}}} \leq C|u - u(x_0)|_{L^\infty(B_{d_x}(x))} \leq C d_x^{\frac{\alpha}{2}} \|\varphi\|_{C^\alpha(\partial B_1)}.$$

Hence we obtain

$$|u(x) - u(y)| \leq C|x - y|^{\frac{\alpha}{2}} \|\varphi\|_{C^\alpha(\partial B_1)}.$$

Assume now that $d_y \leq d_x \leq 2|x - y|$. Then by (1) again we have

$$\begin{aligned}
|u(x) - u(y)| &\leq |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\
&\leq C(d_x^{\frac{\alpha}{2}} + |x_0 - y_0|^{\frac{\alpha}{2}} + d_y^{\frac{\alpha}{2}})\|\varphi\|_{C^\alpha(\partial B_1)} \\
&\leq C|x - y|^{\frac{\alpha}{2}} \|\varphi\|_{C^\alpha(\partial B_1)}
\end{aligned}$$

since $|x_0 - y_0| \leq d_x + |x - y| + d_y \leq 5|x - y|$.

In order to prove (1) we assume $B_1 = B_1((1, 0, \cdots, 0))$, $x_0 = 0$ and $\varphi(0) = 0$. Define $K = \sup_{x \in \partial B_1} |\varphi(x)|/|x|^\alpha$. Note $|x|^2 = 2x_1$ for $x \in \partial B_1$. Therefore for $x \in \partial B_1$ there holds

$$\varphi(x) \leq K|x|^\alpha \leq 2^{\frac{\alpha}{2}} K x_1^{\frac{\alpha}{2}}.$$

Define $v(x) = 2^{\alpha/2} K x_1^{\alpha/2}$ in $B_1$. Then we have

$$\triangle v(x) = 2^{\frac{\alpha}{2}} K \cdot \frac{\alpha}{2}(\frac{\alpha}{2} - 1) x_1^{\frac{\alpha}{2} - 2} < 0 \text{ in } B_1.$$

Theorem 3.1 implies

$$u(x) \leq v(x) = 2^{\frac{\alpha}{2}} K x_1^{\frac{\alpha}{2}} \leq 2^{\frac{\alpha}{2}} K |x|^{\frac{\alpha}{2}} \text{ for any } x \in B_1.$$

Considering $-u$ similarly we get

$$|u(x)| \leq 2^{\frac{\alpha}{2}} K |x|^{\frac{\alpha}{2}} \text{ for any } x \in B_1.$$

This proves (1).

## §4. Energy Method

In this section we discuss harmonic functions by using energy method. In general we assume throughout this section that $a_{ij} \in C(B_1)$ satisfies

$$\lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \text{ for any } x \in B_1 \text{ and } \xi \in \mathbb{R}^n$$

for some positive constants $\lambda$ and $\Lambda$. We consider the function $u \in C^1(B_1)$ satisfying

$$\int_{B_1} a_{ij} D_i u D_j \varphi = 0 \text{ for any } \varphi \in C_0^1(B_1).$$

It is easy to check by integration by parts that harmonic functions satisfy above equation for $a_{ij} = \delta_{ij}$.

**Lemma 4.1 (Cacciopolli Inequality).** *Suppose $u \in C^1(B_1)$ satisfies*

$$\int_{B_1} a_{ij} D_i u D_j \varphi = 0 \textit{ for any } \varphi \in C_0^1(B_1).$$

*Then for any function $\eta \in C_0^1(B_1)$, we have*

$$\int_{B_1} \eta^2 |Du|^2 \leq C \int_{B_1} |D\eta|^2 u^2$$

*where $C$ is a positive constant depending only on $\lambda$ and $\Lambda$.*

*Proof.* For any $\eta \in C_0^1(B_1)$ set $\varphi = \eta^2 u$. Then we have

$$\lambda \int_{B_1} \eta^2 |Du|^2 \leq \Lambda \int_{B_1} \eta |u| |D\eta| |Du|.$$

We obtain the result by Hölder inequality.

**Corollary 4.2.** *Let $u$ be in Lemma 4.1. Then for any $0 \le r < R \le 1$ there holds*

$$\int_{B_r} |Du|^2 \le \frac{C}{(R-r)^2} \int_{B_R} u^2$$

*where $C$ is a positive constant depending only on $\lambda$ and $\Lambda$.*

*Proof.* Take $\eta$ such that $\eta = 1$ on $B_r$, $\eta = 0$ outside $B_R$ and $|D\eta| \le 2(R-r)^{-1}$.

**Corollary 4.3.** *Let $u$ be in Lemma 4.1. Then for any $0 < R \le 1$ there hold*

$$\int_{B_{\frac{R}{2}}} u^2 \le \theta \int_{B_R} u^2$$

*and*

$$\int_{B_{\frac{R}{2}}} |Du|^2 \le \theta \int_{B_R} |Du|^2$$

*where $\theta = \theta(n, \lambda, \Lambda) \in (0, 1)$.*

*Proof.* Take $\eta \in C_0^1(B_R)$ with $\eta = 1$ on $B_{R/2}$ and $|D\eta| \le 2R^{-1}$. Then Lemma 4.1 yields

$$\int_{B_R} |D(\eta u)|^2 \le C \int_{B_R} |D\eta|^2 u^2 \le \frac{C}{R^2} \int_{B_R \setminus B_{\frac{R}{2}}} u^2$$

by noting $D\eta = 0$ in $B_{R/2}$. Hence by Poincaré inequality we get

$$\int_{B_R} (\eta u)^2 \le c(n) R^2 \int_{B_R} |D(\eta u)|^2.$$

Therefore we obtain

$$\int_{B_{\frac{R}{2}}} u^2 \le C \int_{B_R \setminus B_{\frac{R}{2}}} u^2,$$

which implies

$$(C+1) \int_{B_{\frac{R}{2}}} u^2 \le C \int_{B_R} u^2.$$

For the second inequality, observe that Lemma 4.1 holds for $u - a$ for arbitrary constant $a$. Then as before we have

$$\int_{B_R} \eta^2 |Du|^2 \le C \int_{B_R} |D\eta|^2 (u-a)^2 \le \frac{C}{R^2} \int_{B_R \setminus B_{\frac{R}{2}}} (u-a)^2.$$

Poincaré inequality implies with $a = |B_R \setminus B_{\frac{R}{2}}|^{-1} \int_{B_R \setminus B_{\frac{R}{2}}} u$

$$\int_{B_R \setminus B_{\frac{R}{2}}} (u-a)^2 \le c(n) R^2 \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2.$$

Hence we obtain

$$\int_{B_{\frac{R}{2}}} |Du|^2 \le C \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2$$

in particular

$$(C+1) \int_{B_{\frac{R}{2}}} |Du|^2 \le C \int_{B_R} |Du|^2.$$

*Remark.* Corollary 4.3 implies, in particular, that a harmonic function in $\mathbb{R}^n$ with finite $L^2$-norm is identically zero and that a harmonic function in $\mathbb{R}^n$ with finite Dirichlet integral is constant.

*Remark.* By iterating the result in Corollary 4.3, we have the following estimates. Let $u$ be in Lemma 4.1. Then for any $0 < \rho < r \le 1$ there hold

$$\int_{B_\rho} u^2 \le C(\frac{\rho}{r})^\mu \int_{B_r} u^2$$

and

$$\int_{B_\rho} |Du|^2 \le C(\frac{\rho}{r})^\mu \int_{B_r} |Du|^2$$

for some positive constant $\mu = \mu(n, \lambda, \Lambda)$. Later on we will prove that we can take $\mu \in (n-2, n)$ in the second inequality. For harmonic functions we have better results.

**Lemma 4.4.** *Suppose $\{a_{ij}\}$ is a constant positive definite matrix with*

$$\lambda |\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda |\xi|^2 \quad \textit{for any} \quad \xi \in \mathbb{R}^n$$

*for some constants $0 < \lambda \le \Lambda$. Suppose $u \in C^1(B_1)$ satisfies*

(1)
$$\int_{B_1} a_{ij} D_i u D_j \varphi = 0 \textit{ for any } \varphi \in C_0^1(B_1).$$

*Then for any $0 < \rho \le r$, there holds*

(2)
$$\int_{B_\rho} |u|^2 \le c \left(\frac{\rho}{r}\right)^n \int_{B_r} |u|^2$$

*and*

$$(3) \qquad \int_{B_\rho} |u-u_\rho|^2 \le c\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |u-u_r|^2$$

*where $c = c(\lambda, \Lambda)$ is a positive constant and $u_r$ denotes the average of $u$ in $B_r$.*

*Proof.* By dilation, consider $r = 1$. We restrict our consideration to the range $\rho \in \big(0, \frac{1}{2}\big]$, since (2) and (3) are trivial for $\rho \in (\frac{1}{2}, 1]$.

**Claim.**

$$|u|^2_{L^\infty(B_{\frac{1}{2}})} + |Du|^2_{L^\infty(B_{\frac{1}{2}})} \le c(\lambda, \Lambda) \int_{B_1} |u|^2.$$

Therefore for $\rho \in \big(0, \frac{1}{2}\big]$

$$\int_{B_\rho} |u|^2 \le \rho^n |u|^2_{L^\infty(B_{\frac{1}{2}})} \le c\rho^n \int_{B_1} |u|^2$$

and

$$\int_{B_\rho} |u-u_\rho|^2 \le \int_{B_\rho} |u-u(0)|^2 \le \rho^{n+2} |Du|^2_{L^\infty(B_{\frac{1}{2}})} \le c\rho^{n+2} \int_{B_1} |u|^2.$$

If $u$ is a solution of (1), so is $u - u_1$. With $u$ replaced by $u - u_1$ in the above inequality, there holds

$$\int_{B_\rho} |u-u_\rho|^2 \le c\rho^{n+2} \int_{B_1} |u-u_1|^2.$$

*Proof of Claim.* **Method 1.** By rotation, we may assume $\{a_{ij}\}$ is a diagonal matrix. Hence (1) becomes

$$\sum_{i=1}^n \lambda_i D_{ii} u = 0$$

with $0 < \lambda \le \lambda_i \le \Lambda$ for $i = 1, \cdots n$. It is easy to see there exists an $r_0 = r_0(\lambda, \Lambda) \in \big(0, \frac{1}{2}\big)$ such that for any $x_0 \in B_{\frac{1}{2}}$ the rectangle

$$\left\{ x; \frac{|x_i - x_{0i}|}{\sqrt{\lambda_i}} < r_0 \right\}$$

is contained in $B_1$. Change the coordinate

$$x_i \longmapsto y_i = \frac{x_i}{\sqrt{\lambda_i}}$$

and set

$$v(y) = u(x).$$

Then $v$ is harmonic in $\{y; \sum_{i=1}^n \lambda_i y_i^2 < 1\}$. In the ball $\{y; |y - y_0| < r_0\}$ use the interior estimates to yield

$$|v(y_0)|^2 + |Dv(y_0)|^2 \leq c(\lambda, \Lambda) \int_{B_{r_0}(y_0)} v^2 \leq c(\lambda, \Lambda) \int_{\{\sum_{i=1}^n \lambda_i y_i^2 < 1\}} v^2.$$

Transform back to $u$ to get

$$|u(x_0)|^2 + |Du(x_0)|^2 \leq c(\lambda, \Lambda) \int_{|x|<1} u^2.$$

**Method 2.** If $u$ is a solution to (1), so are any derivatives of $u$. By applying Corollary 4.2 to derivatives of $u$ we conclude that for any positive integer $k$

$$\|u\|_{H^k(B_{\frac{1}{2}})} \leq c(k, \lambda, \Lambda) \|u\|_{L^2(B_1)}.$$

If we fix a value of $k$ sufficiently large with respect to $n$, $H^k(B_{\frac{1}{2}})$ is continuously embedded into $C^1(\bar{B}_{\frac{1}{2}})$ and therefore

$$|u|_{L^\infty(B_{\frac{1}{2}})} + |Du|_{L^\infty(B_{\frac{1}{2}})} \leq c(\lambda, \Lambda) \|u\|_{L^2(B_1)}.$$

This finishes the proof.

# CHAPTER 2
# MAXIMUM PRINCIPLES

## GUIDE

Most statements in section 2.1 are rather simple.One probabilly needs to go over Theorm 1.8 and Proposition 1.9.Section 2.2 is often the starting point of the A Priori estimates.Section 2.4 can be omited in the first reading as we will look at it again in section 5.1.The moving plane method explained in section 2.5 has many recent applications.We choice a very simple example to illustare such method.The result goes back to Gidas-Ni-Nirenberg, but the proof contains some recent observations in the paper [BNV].The classical paper of Gilbarg-Serrin [GS] may be a very good adding for this chapter. It may be also a good idea to assume the Hanack Inequality of Krylov- Safanov in section 5.2 to ask students to improve some of the results in the paper [GS].

In this chapter we will discuss maximum principles and their applications. Two kinds of maximum principles will be discussed. One is due to Hopf and the other to Alexandroff. The former gives the estimates of solutions in terms of the $L^\infty$-norm of the nonhomogenous terms while the latter gives the estimates in terms of the $L^n$-norm. Applications include various a priori estimates and moving plane method.

## §1. Strong Maximum Principle

Suppose $\Omega$ is a bounded and connected domain in $\mathbb{R}^n$. Consider the operator $L$ in $\Omega$

$$Lu \equiv a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u$$

for $u \in C^2(\Omega) \cap C(\bar{\Omega})$. We always assume that $a_{ij}$, $b_i$ and $c$ are continuous and hence bounded in $\bar{\Omega}$ and that $L$ is uniformly elliptic in $\Omega$ in the following sense

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n$$

for some positive constant $\lambda$.

**Lemma 1.1.** *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu > 0$ in $\Omega$ with $c(x) \leq 0$ in $\Omega$. If $u$ has a nonnegative maximum in $\bar{\Omega}$, then $u$ cannot attain this maximum in $\Omega$.*

*Proof.* Suppose $u$ attains its nonnegative maximum of $\bar{\Omega}$ in $x_0 \in \Omega$. Then $D_iu(x_0) = 0$ and the matrix $B = (D_{ij}(x_0))$ is semi-negative definite. By ellipticity condition the matrix $A = (a_{ij}(x_0))$ is positive definite. Hence the matrix $AB$ is semi-negative definite with a nonpositive trace, i.e., $a_{ij}(x_0)D_{ij}u(x_0) \leq 0$. This implies $Lu(x_0) \leq 0$, which is a contradiction.

*Remark.* If $c(x) \equiv 0$, then the requirement for nonnegativeness can be removed. This remark also holds for some results in the rest of this section.

**Theorem 1.2 (Weak Maximum Principle).** *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in $\Omega$ with $c(x) \leq 0$ in $\Omega$. Then $u$ attains on $\partial\Omega$ its nonnegative maximum in $\bar{\Omega}$.*

*Proof.* For any $\varepsilon > 0$, consider $w(x) = u(x) + \varepsilon e^{\alpha x_1}$ with $\alpha$ to be determined. Then we have

$$Lw = Lu + \varepsilon e^{\alpha x_1}(a_{11}\alpha^2 + b_1\alpha + c).$$

Since $b_1$ and $c$ are bounded and $a_{11}(x) \geq \lambda > 0$ for any $x \in \Omega$, by choosing $\alpha > 0$ large enough we get

$$a_{11}(x)\alpha^2 + b_1(x)\alpha + c(x) > 0 \text{ for any } x \in \Omega.$$

This implies $Lw > 0$ in $\Omega$. By Lemma 1.1, $w$ attains its nonnegative maximum only on $\partial\Omega$, i.e.,

$$\sup_{\Omega} w \leq \sup_{\partial\Omega} w^+.$$

Then we obtain

$$\sup_{\Omega} u \leq \sup_{\Omega} w \leq \sup_{\partial\Omega} w^+ \leq \sup_{\partial\Omega} u^+ + \varepsilon \sup_{x \in \partial\Omega} e^{\alpha x_1}.$$

We finish the proof by letting $\varepsilon \to 0$.

As an application we have the uniqueness of solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ to the following Dirichlet boundary value problem for $f \in C(\Omega)$ and $\varphi \in C(\partial\Omega)$

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= \varphi \text{ on } \partial\Omega \end{aligned}$$

if $c(x) \leq 0$ in $\Omega$.

*Remark.* The boundedness of domain $\Omega$ is essential, since it guarantees the existence of maximum and minimum of $u$ in $\bar{\Omega}$. The uniqueness does not hold if the domain is unbounded. Some examples are given in Section 1 in Chapter 1. Equally important is the nonpositiveness of the coefficient $c$.

*Example.* Set $\Omega = \{(x, y) \in \mathbb{R}^2; 0 < x < \pi, 0 < y < \pi\}$. Then $u = \sin x \sin y$ is a nontrivial solution for the problem

$$\begin{aligned} \triangle u + 2u &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

**Theorem 1.3 (Hopf Lemma).** *Let $B$ be an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$. Suppose $u \in C^2(B) \cap C(B \cup \{x_0\})$ satisfies $Lu \geq 0$ in $B$ with $c(x) \leq 0$ in $B$. Assume in addition that*

$$u(x) < u(x_0) \text{ for any } x \in B \text{ and } u(x_0) \geq 0.$$

*Then for each outward direction $\vec{\nu}$ at $x_0$ with $\vec{\nu} \cdot \vec{n}(x_0) > 0$ there holds*

$$\liminf_{t \to 0^+} \frac{1}{t}[u(x_0) - u(x_0 - t\vec{\nu})] > 0.$$

*Remark.* If in addition $u \in C^1(B \cup \{x_0\})$, then we have

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

*Proof.* We may assume that $B$ has the center at the origin with radius $r$. We assume further that $u \in C(\bar{B})$ and $u(x) < u(x_0)$ for any $x \in \bar{B} \setminus \{x_0\}$ (since we can construct a tangent ball $B_1$ to $B$ at $x_0$ and $B_1 \subset B$).

Consider $v(x) = u(x) + \varepsilon h(x)$ for some nonnegative function $h$. We will choose $\varepsilon > 0$ appropriately such that $v$ attains its nonnegative maximum only at $x_0$. Denote $\Sigma = B \cap B_{\frac{1}{2}r}(x_0)$. Define $h(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2}$ with $\alpha$ to be determined. We check in the following that

$$Lh > 0 \text{ in } \Sigma.$$

Direct calculation yields

$$\begin{aligned} Lh &= e^{-\alpha|x|^2}\big\{4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x)x_i x_j - 2\alpha \sum_{i=1}^{n} a_{ii}(x) - 2\alpha \sum_{n=1}^{n} b_i(x)x_i + c\big\} - ce^{-\alpha r^2} \\ &\geq e^{-\alpha|x|^2}\big\{4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x)x_i x_j - 2\alpha \sum_{i=1}^{n}[a_{ii}(x) + b_i(x)x_i] + c\big\}. \end{aligned}$$

By ellipticity assumption, we have

$$\sum_{i,j=1}^{n} a_{ij}(x)x_i x_j \geq \lambda|x|^2 \geq \lambda\left(\frac{r}{2}\right)^2 > 0 \text{ in } \Sigma.$$

So for $\alpha$ large enough, we conclude $Lh > 0$ in $\Sigma$. With such $h$, we have $Lv = Lu + \varepsilon Lh > 0$ in $\Sigma$ for any $\varepsilon > 0$. By Lemma 1.1, $v$ cannot attain its nonnegative maximum inside $\Sigma$.

Next we prove for some small $\varepsilon > 0$ $v$ attains at $x_0$ its nonnegative maximum. Consider $v$ on the boundary $\partial\Sigma$.

(i) For $x \in \partial\Sigma \cap B$, since $u(x) < u(x_0)$, so $u(x) < u(x_0) - \delta$ for some $\delta > 0$. Take $\varepsilon$ small such that $\varepsilon h < \delta$ on $\partial\Sigma \cap B$. Hence for such $\varepsilon$ we have $v(x) < u(x_0)$ for $x \in \partial\Sigma \cap B$.

(ii) On $\Sigma \cap \partial B$, $h(x) = 0$ and $u(x) < u(x_0)$ for $x \neq x_0$. Hence $v(x) < u(x_0)$ on $\Sigma \cap \partial B \setminus \{x_0\}$ and $v(x_0) = u(x_0)$.

Therefore we conclude

$$\frac{v(x_0) - v(x_0 - t\nu)}{t} \geq 0 \text{ for any small } t > 0.$$

Hence we obtain by letting $t \to 0$

$$\liminf_{t \to 0} \frac{1}{t}[u(x_0) - u(x_0 - t\nu)] \geq -\varepsilon \frac{\partial h}{\partial \nu}(x_0).$$

By definition of $h$, we have

$$\frac{\partial h}{\partial \nu}(x_0) = \frac{\partial h}{\partial n}(x_0)\vec{n} \cdot \vec{\nu} = -2\alpha r e^{-\alpha r^2} \vec{n} \cdot \vec{\nu} < 0.$$

This finishes the proof.

**Theorem 1.4 (Strong Maximum Principle).** *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$ with $c(x) \leq 0$ in $\Omega$. Then the nonnegative maximum of $u$ in $\bar{\Omega}$ can be assumed only on $\partial\Omega$ unless $u$ is a constant.*

*Proof.* Let $M$ be the nonnegative maximum of $u$ in $\bar{\Omega}$. Set $\Sigma = \{x \in \Omega; u(x) = M\}$. It is relatively closed in $\Omega$. We need to show $\Sigma = \Omega$.

We prove by contradiction. If $\Sigma$ is a proper subset of $\Omega$, then we may find an open ball $B \subset \Omega \setminus \Sigma$ with a point on its boundary belonging to $\Sigma$. (In fact we may choose a point $p \in \Omega \setminus \Sigma$ such that $d(p, \Sigma) < d(p, \partial\Omega)$ first and then extend the ball centered at $p$. It hits $\Sigma$ before hitting $\partial\Omega$.) Suppose $x_0 \in \partial B \cap \Sigma$. Obviously we have $Lu \geq 0$ in $B$ and

$$u(x) < u(x_0) \text{ for any } x \in B \quad \text{and } u(x_0) = M \geq 0.$$

Theorem 1.3 implies $\frac{\partial u}{\partial n}(x_0) > 0$ where $\vec{n}$ is the outward normal direction at $x_0$ to the ball $B$. While $x_0$ is the interior maximal point of $\Omega$, hence $Du(x_0) = 0$. This leads to a contradiction.

**Corollary 1.5 (Comparison Principle).** *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in $\Omega$ with $c(x) \leq 0$ in $\Omega$. If $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in $\Omega$. In fact, either $u < 0$ in $\Omega$ or $u \equiv 0$ in $\Omega$.*

In order to discuss the boundary value problem with general boundary condition, we need the following result, which is just a corollary of Theorem 1.3 and Theorem 1.4.

**Corollary 1.6.** *Suppose $\Omega$ has the interior sphere property and that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies $Lu \geq 0$ in $\Omega$ with $c(x) \leq 0$. Assume $u$ attains its nonnegative maximum at $x_0 \in \bar{\Omega}$. Then $x_0 \in \partial\Omega$ and for any outward direction $\nu$ at $x_0$ to $\partial\Omega$*

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

*unless $u$ is constant in $\bar{\Omega}$.*

*Application.* Suppose $\Omega$ is bounded in $\mathbb{R}^n$ and satisfies the interior sphere property. Consider the the following boundary value problem

$$(*) \qquad \begin{aligned} Lu =& f \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u =& \varphi \quad \text{on } \partial\Omega \end{aligned}$$

for some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$. Assume in addition that $c(x) \leq 0$ in $\Omega$ and $\alpha(x) \geq 0$ on $\partial\Omega$. Then the problem (*) has a unique solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ if $c \not\equiv 0$ or $\alpha \not\equiv 0$. If $c \equiv 0$ and $\alpha \equiv 0$, the problem (*) has unique solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ up to additive constants.

*Proof.* Suppose $u$ is a solution to the following homogeneous equation

$$\begin{aligned} Lu = 0 &\quad \text{in } \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u = 0 &\quad \text{on } \partial\Omega. \end{aligned}$$

Case 1. $c \not\equiv 0$ or $\alpha \not\equiv 0$. We want to show $u \equiv 0$.

Suppose that $u$ has a positive maximum at $x_0 \in \bar{\Omega}$. If $u \equiv \mathit{const.} > 0$, this contradicts the condition $c \not\equiv 0$ in $\Omega$ or $\alpha \not\equiv 0$ on $\partial\Omega$. Otherwise $x_0 \in \partial\Omega$ and $\frac{\partial u}{\partial n}(x_0) > 0$ by Corollary 1.6, which contradicts the boundary value. Therefore $u \equiv 0$.

Case 2. $c \equiv 0$ and $\alpha \equiv 0$. We want to show $u \equiv$ const.

Suppose $u$ is a nonconstant solution. Then its maximum in $\bar{\Omega}$ is assumed only on $\partial\Omega$ by Theorem 1.4, say at $x_0 \in \partial\Omega$. Again Corollary 1.6 implies $\frac{\partial u}{\partial n}(x_0) > 0$ . This is a contradiction.

The following theorem, due to Serrin, generalizes the comparison principle under no restriction on $c(x)$.

**Theorem 1.7.** *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$. If $u \leq 0$ in $\Omega$, then either $u < 0$ in $\Omega$ or $u \equiv 0$ in $\Omega$.*

*Proof.* Method 1. Suppose $u(x_0) = 0$ for some $x_0 \in \Omega$. We will prove that $u \equiv 0$ in $\Omega$.

Write $c(x) = c^+(x) - c^-(x)$ where $c^+(x)$ and $c^-(x)$ are the positive part and negative part of $c(x)$ respectively. Hence $u$ satisfies

$$a_{ij}D_{ij}u + b_iD_iu - c^-u \geq -c^+u \geq 0.$$

So we have $u \equiv 0$ by Theorem 1.4.

Method 2. Set $v = ue^{-\alpha x_1}$ for some $\alpha > 0$ to be determined. By $Lu \geq 0$, we have

$$a_{ij}D_{ij}v + [\alpha(a_{1i} + a_{i1}) + b_i]D_iv + (a_{11}\alpha^2 + b_1\alpha + c)v \geq 0.$$

Choose $\alpha$ large enough such that $a_{11}\alpha^2 + b_1\alpha + c > 0$. Therefore $v$ satisfies

$$a_{ij}D_{ij}v + [\alpha(a_{1i} + a_{i1}) + b_i]D_iv \geq 0.$$

Hence we apply Theorem 1.4 to $v$ to conclude that either $v < 0$ in $\Omega$ or $v \equiv 0$ in $\Omega$.

Next result is the general maximum principle for the operator $L$ with no restriction on $c(x)$.

**Theorem 1.8.** *Suppose there exists a $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying $w > 0$ in $\bar{\Omega}$ and $Lw \leq 0$ in $\Omega$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in $\Omega$, then $\frac{u}{w}$ cannot assume in $\Omega$ its nonnegative maximum unless $\frac{u}{w} \equiv$ const. If, in addition, $\frac{u}{w}$ assumes its nonnegative maximum at $x_0 \in \partial\Omega$ and $\frac{u}{w} \not\equiv$ const., then for any outward direction $\nu$ at $x_0$ to $\partial\Omega$ there holds*

$$\frac{\partial}{\partial \nu}\left(\frac{u}{w}\right)(x_0) > 0$$

*if $\partial\Omega$ has the interior sphere property at $x_0$.*

*Proof.* Set $v = \frac{u}{w}$. Then $v$ satisfies

$$a_{ij}D_{ij}v + B_iD_iv + (\frac{Lw}{w})v \geq 0$$

where $B_i = b_i + \frac{2}{w}a_{ij}D_{ij}w$. We may apply Theorem 1.4 and Corollary 1.6 to $v$.

*Remark.* If the operator $L$ in $\Omega$ satisfies the condition of Theorem 1.8, then $L$ has the comparison principle. In particular, the Dirichlet boundary value problem

$$Lu = f \text{ in } \Omega$$
$$u = \varphi \text{ on } \partial\Omega$$

has at most one solution.

Next result is the so-called maximum principle for *narrow domain.*

**Proposition 1.9.** *Let $d$ be a positive number and $\vec{e}$ be a unit vector such that $|(y - x) \cdot \vec{e}| < d$ for any $x, y \in \Omega$. Then there exists a $d_0 > 0$, depending only on $\lambda$ and the sup-norm of $b_i$ and $c^+$, such that the assumptions of Theorem 1.8 are satisfied if $d \leq d_0$.*

*Proof.* By choosing $\vec{e} = (1, 0, \cdots, 0)$ we suppose $\bar{\Omega}$ lies in $\{0 < x_1 < d\}$. Assume in addition $|b_i|, c^+ \leq N$ for some positive constant $N$. We construct $w$ as follows. Set $w = e^{\alpha d} - e^{\alpha x_1} > 0$ in $\bar{\Omega}$. By direct calculation we have

$$Lw = -(a_{11}\alpha^2 + b_1\alpha)e^{\alpha x_1} + c(e^{\alpha d} - e^{\alpha x_1}) \leq -(a_{11}\alpha^2 + b_1\alpha) + Ne^{\alpha d}.$$

Choose $\alpha$ so large that

$$a_{11}\alpha^2 + b_1\alpha \geq \lambda\alpha^2 - N\alpha \geq 2N.$$

Hence $Lw \leq -2N + Ne^{\alpha d} = N(e^{\alpha d} - 2) \leq 0$ if $d$ is small such that $e^{\alpha d} \leq 2$.

*Remark.* Some results in this section, including Proposition 1.9, hold for unbounded domain. Compare Proposition 1.9 with Theorem 4.8.

## §2. A Priori Estimates

In this section we derive a priori estimates for solutions to Dirichlet problem and Neumann problem.

Suppose $\Omega$ is a bounded and connected domain in $\mathbb{R}^n$. Consider the operator $L$ in $\Omega$

$$Lu \equiv a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u$$

for $u \in C^2(\Omega) \cap C(\bar{\Omega})$. We assume that $a_{ij}$, $b_i$ and $c$ are continuous and hence bounded in $\bar{\Omega}$ and that $L$ is uniformly elliptic in $\Omega$, i.e.,

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n$$

where $\lambda$ is a positive number. We denote by $\Lambda$ the sup-norm of $a_{ij}$ and $b_i$, i.e.,

$$\max_{\Omega}|a_{ij}| + \max_{\Omega}|b_i| \leq \Lambda.$$

**Proposition 2.1.** *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

*for some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$. If $c(x) \leq 0$, then there holds*

$$|u(x)| \leq \max_{\partial\Omega}|\varphi| + C\max_{\Omega}|f| \text{ for any } x \in \Omega$$

*where $C$ is a positive constant depending only on $\lambda$, $\Lambda$ and $diam(\Omega)$.*

*Proof.* We will construct a function $w$ in $\Omega$ such that
(i) $L(w \pm u) = Lw \pm f \leq 0$, or $Lw \leq \mp f$ in $\Omega$;
(ii) $w \pm u = w \pm \varphi \geq 0$, or $w \geq \mp\varphi$ on $\partial\Omega$.
Denote $F = \max_{\Omega}|f|$ and $\Phi = \max_{\partial\Omega}|\varphi|$. We need

$$\begin{aligned} Lw &\leq -F \text{ in } \Omega \\ w &\geq \Phi \text{ on } \partial\Omega. \end{aligned}$$

Suppose the domain $\Omega$ lies in the set $\{0 < x_1 < d\}$ for some $d > 0$. Set $w = \Phi + (e^{\alpha d} - e^{\alpha x_1})F$ with $\alpha > 0$ to be chosen later. Then we have by direct calculation

$$\begin{aligned} -Lw &= (a_{11}\alpha^2 + b_1\alpha)Fe^{\alpha x_1} - c\Phi - c(e^{\alpha d} - e^{\alpha x_1})F \\ &\geq (a_{11}\alpha^2 + b_1\alpha)F \geq (\alpha^2\lambda + b_1\alpha)F \geq F \end{aligned}$$

by choosing $\alpha$ large such that $\alpha^2\lambda + b_1(x)\alpha \geq 1$ for any $x \in \Omega$. Hence $w$ satisfies (i) and (ii). By Corollary 1.5 (the comparison principle) we conclude $-w \leq u \leq w$ in $\Omega$, in particular

$$\sup_{\Omega} |u| \leq \Phi + (e^{\alpha d} - 1)F$$

where $\alpha$ is a positive constant depending only on $\lambda$ and $\Lambda$.

**Proposition 2.2.** *Suppose $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\begin{cases} Lu = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u = \varphi & \text{on } \partial\Omega \end{cases}$$

*where $\vec{n}$ is the outward normal direction to $\partial\Omega$. If $c(x) \leq 0$ in $\Omega$ and $\alpha(x) \geq \alpha_0 > 0$ on $\partial\Omega$, then there holds*

$$|u(x)| \leq C\{\max_{\partial\Omega}|\varphi| + \max_{\Omega}|f|\} \text{ for any } x \in \Omega$$

*where $C$ is a positive constant depending only on $\lambda$, $\Lambda$, $\alpha_0$ and $diam(\Omega)$.*

*Proof.* Special case: $c(x) \leq -c_0 < 0$. We will show

$$|u(x)| \leq \frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi \quad \text{for any } x \in \Omega.$$

Define $v = \frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi \pm u$. Then we have

$$Lv = c(x)\left(\frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi\right) \pm f \leq -F \pm f \leq 0 \text{ in } \Omega$$
$$\frac{\partial v}{\partial n} + \alpha v = \alpha\left(\frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi\right) \pm \varphi \geq \Phi \pm \varphi \geq 0 \text{ on } \partial\Omega.$$

If $v$ has a negative minimum in $\bar{\Omega}$, then $v$ attains it on $\partial\Omega$ by Theorem 1.2, say at $x_0 \in \partial\Omega$. This implies $\frac{\partial v}{\partial n}(x_0) \leq 0$ for $\vec{n} = \vec{n}(x_0)$, the outward normal direction at $x_0$. Therefore we get

$$\left(\frac{\partial v}{\partial n} + \alpha v\right)(x_0) \leq \alpha v(x_0) < 0$$

which is a contradiction. Hence we have $v \geq 0$ in $\bar{\Omega}$, in particular,

$$|u(x)| \leq \frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi \text{ for any } x \in \Omega.$$

Note that for this special case $c_0$ and $\alpha_0$ are independent of $\lambda$ and $\Lambda$.

General Case: $c(x) \leq 0$ for any $x \in \Omega$.

Consider the auxiliary function $u(x) = z(x)w(x)$ where $z$ is a positive function in $\bar{\Omega}$ to be determined. Direct calculation shows that $w$ satisfies

$$a_{ij}D_{ij}w + B_iD_iw + \left(c + \frac{a_{ij}D_{ij}z + b_iD_iz}{z}\right)w = \frac{f}{z} \text{ in } \Omega$$
$$\frac{\partial w}{\partial n} + \left(\alpha + \frac{1}{z}\frac{\partial z}{\partial n}\right)w = \frac{\varphi}{z} \text{ on } \partial\Omega$$

where $B_i = \frac{1}{z}(a_{ij} + a_{ji})D_jz + b_i$. We need to choose the function $z > 0$ in $\bar{\Omega}$ such that there hold in

$$c + \frac{a_{ij}D_{ij}z + b_iD_iz}{z} \leq -c_0(\lambda, \Lambda, d, \alpha_0) < 0 \quad \text{in } \Omega$$
$$\alpha + \frac{1}{z}\frac{\partial z}{\partial n} \geq \frac{1}{2}\alpha_0 \quad \text{on } \partial\Omega,$$

or

$$\frac{a_{ij}D_{ij}z + b_iD_iz}{z} \leq -c_0 < 0 \quad \text{in } \Omega$$
$$\left|\frac{1}{z}\frac{\partial z}{\partial n}\right| \leq \frac{1}{2}\alpha_0 \quad \text{on } \partial\Omega.$$

Suppose the domain $\Omega$ lies in $\{0 < x_1 < d\}$. Choose $z(x) = A + e^{\beta d} - e^{\beta x_1}$ for $x \in \Omega$ for some positive $A$ and $\beta$ to be determined. Direct calculation shows

$$-\frac{1}{z}\left(a_{ij}D_{ij}z + b_i D_i z\right) = \frac{(\beta^2 a_{11} + \beta b_1)e^{\beta x_1}}{A + e^{\beta d} - e^{\beta x_1}} \geq \frac{\beta^2 a_{11} + \beta b_1}{A + e^{\beta d}} \geq \frac{1}{A + e^{\beta d}} > 0,$$

if $\beta$ is chosen such that $\beta^2 a_{11} + \beta b_1 \geq 1$. Then we have

$$\left|\frac{1}{z}\frac{\partial z}{\partial n}\right| \leq \frac{\beta}{A}e^{\beta d} \leq \frac{1}{2}\alpha_0$$

if $A$ is chosen large. This reduces to the special case we just discussed. The new extra first order term does not change the result. We may apply the special case to $w$.

*Remark.* The result fails if we just assume $\alpha(x) \geq 0$ on $\partial\Omega$. In fact, we cannot even get the uniqueness.

## §3. Gradient Estimates

The basic idea in the treatment of gradient estimates, due to Bernstein, involves differentiation of the equation with respect to $x_k$, $k = 1, \ldots, n$, followed by multiplication by $D_k u$ and summation over $k$. The maximum principle is then applied to the resulting equation in the function $v = |Du|^2$, possibly with some modification. There are two kinds of gradient estimates, global gradient estimates and interior gradient estimates. We will use semi-linear equations to illustrate the idea.

Suppose $\Omega$ is a bounded and connected domain in $\mathbb{R}^n$. Consider the equation

$$a_{ij}(x)D_{ij}u + b_i(x)D_i u = f(x,u) \quad \text{in } \Omega$$

for $u \in C^2(\Omega)$ and $f \in C(\Omega \times \mathbb{R})$. We always assume that $a_{ij}$ and $b_i$ are continuous and hence bounded in $\bar{\Omega}$ and that the equation is uniformly elliptic in $\Omega$ in the following sense

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n$$

for some positive constant $\lambda$.

**Proposition 3.1.** *Suppose $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$(1) \qquad a_{ij}(x)D_{ij}u + b_i(x)D_i u = f(x,u) \quad \textit{in } \Omega$$

*for $a_{ij}, b_i \in C^1(\bar{\Omega})$ and $f \in C^1(\bar{\Omega} \times \mathbb{R})$. Then there holds*

$$\sup_{\Omega}|Du| \leq \sup_{\partial\Omega}|Du| + C$$

*where $C$ is a positive constant depending only on $\lambda$, diam$(\Omega)$, $|a_{ij}, b_i|_{C^1(\bar{\Omega})}$, $M = |u|_{L^\infty(\Omega)}$ and $|f|_{C^1(\bar{\Omega}\times[-M,M])}$.*

*Proof.* Set $L \equiv a_{ij}D_{ij} + b_iD_i$. We calculate $L(|Du|^2)$ first. Note

$$D_i(|Du|^2) = 2D_kuD_{ki}u$$

and

$$D_{ij}(|Du|^2) = 2D_{ki}D_{kj}u + 2D_kuD_{kij}u. \tag{2}$$

Differentiating (1) with respect to $x_k$, multiplying by $D_ku$ and summing over $k$, we have by (2)

$$\begin{aligned} &a_{ij}D_{ij}(|Du|^2) + b_iD_i(|Du|^2) = 2a_{ij}D_{ki}uD_{kj}u \\ &- 2D_ka_{ij}D_kuD_{ij}u - 2D_kb_iD_kuD_iu + 2D_zf|Du|^2 + 2D_kfD_ku. \end{aligned}$$

Ellipticity assumption implies

$$\sum_{i,j,k} a_{ij}D_{ki}uD_{kj}u \geq \lambda|D^2u|^2.$$

By Cauchy inequality, we have

$$L(|Du|^2) \geq \lambda|D^2u|^2 - C|Du|^2 - C$$

where $C$ is a positive constant depending only on $\lambda$, $|a_{ij}, b_i|_{C^1(\bar{\Omega})}$ and $|f|_{C^1(\bar{\Omega}\times[-M,M])}$. We need to add another term $u^2$. We have by ellipticity assumption

$$\begin{aligned} L(u^2) &= 2a_{ij}D_iuD_ju + 2u\{a_{ij}D_{ij}u + b_iD_iu\} \\ &\geq 2\lambda|Du|^2 + 2uf. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} L(|Du|^2 + \alpha u^2) &\geq \lambda|D^2u|^2 + (2\lambda\alpha - C)|Du|^2 - C \\ &\geq \lambda|D^2u|^2 + |Du|^2 - C \end{aligned}$$

if we choose $\alpha > 0$ large, with $C$ depending in addition on $M$. In order to control the constant term we may consider another function $e^{\beta x_1}$ for $\beta > 0$. Hence we get

$$L\big(|Du|^2 + \alpha u^2 + e^{\beta x_1}\big) \geq \lambda|D^2u|^2 + |Du|^2 + \big\{\beta^2 a_{11}e^{\beta x_1} + \beta b_1e^{\beta x_1} - C\big\}.$$

If we put the domain $\Omega \subset \{x_1 > 0\}$, then $e^{\beta x_1} \geq 1$ for any $x \in \Omega$. By choosing $\beta$ large, we may make the last term positive. Therefore, if we set $w = |Du|^2 + \alpha|u|^2 + e^{\beta x_1}$ for large $\alpha, \beta$ depending only on $\lambda$, $\mathrm{diam}(\Omega)$, $|a_{ij}, b_i|_{C^1(\bar{\Omega})}$, $M = |u|_{L^\infty(\Omega)}$ and $|f|_{C^1(\bar{\Omega}\times[-M,M])}$, then we obtain

$$Lw \geq 0 \quad \text{in } \Omega.$$

By the maximum principle we have

$$\sup_{\Omega} w \leq \sup_{\partial\Omega} w.$$

This finishes the proof.

Similarly, we can discuss the interior gradient bound. In this case, we just require the bound of $\sup_{\Omega}|u|$.

**Proposition 3.2.** *Suppose $u \in C^3(\Omega)$ satisfies*

$$a_{ij}(x)D_{ij}u + b_i(x)D_iu = f(x,u) \quad \textit{in } \Omega$$

*for $a_{ij}, b_i \in C^1(\bar{\Omega})$ and $f \in C^1(\bar{\Omega} \times \mathbb{R})$. Then there holds for any compact subset $\Omega' \subset\subset \Omega$*

$$\sup_{\Omega'}|Du| \leq C$$

*where $C$ is a positive constant depending only on $\lambda$, diam$(\Omega)$, dist$(\Omega', \partial\Omega)$, $|a_{ij}, b_i|_{C^1(\bar{\Omega})}$, $M = |u|_{L^\infty(\Omega)}$ and $|f|_{C^1(\bar{\Omega}\times[-M,M])}$.*

*Proof.* We need to take a cut-off function $\gamma \in C_0^\infty(\Omega)$ with $\gamma \geq 0$ and consider the auxiliary function with the following form

$$w = \gamma|Du|^2 + \alpha|u|^2 + e^{\beta x_1}.$$

Set $v = \gamma|Du|^2$. Then we have for operator $L$ defined as before

$$Lv = (L\gamma)|Du|^2 + \gamma L(|Du|^2) + 2a_{ij}D_i\gamma D_j|Du|^2.$$

Recall an inequality in the proof of Proposition 3.1

$$L(|Du|^2) \geq \lambda|D^2u|^2 - C|Du|^2 - C.$$

Hence we have

$$Lv \geq \lambda\gamma|D^2u|^2 + 2a_{ij}D_kuD_i\gamma D_{kj}u - C|Du|^2 + (L\gamma)|Du|^2 - C.$$

Cauchy inequality implies for any $\varepsilon > 0$

$$|2a_{ij}D_kuD_i\gamma D_{kj}u| \le \varepsilon|D\gamma|^2|D^2u|^2 + c(\varepsilon)|Du|^2.$$

For the cut-off function $\gamma$, we require that

$$|D\gamma|^2 \le C\gamma \text{ in } \Omega.$$

Therefore we have by taking $\varepsilon > 0$ small

$$\begin{aligned} Lv &\ge \lambda\gamma|D^2u|^2\left(1 - \varepsilon\frac{|D\gamma|^2}{\gamma}\right) - C|Du|^2 - C \\ &\ge \frac{1}{2}\lambda\gamma|D^2u|^2 - C|Du|^2 - C. \end{aligned}$$

Now we may proceed as before.

In the rest of this section we use barrier functions to derive the boundary gradient estimates. We need to assume that the domain $\Omega$ satisfies the uniform exterior sphere property.

**Proposition 3.3.** *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$a_{ij}(x)D_{ij}u + b_i(x)D_iu = f(x,u) \quad \textit{in } \Omega$$

*for $a_{ij}, b_i \in C(\bar{\Omega})$ and $f \in C(\bar{\Omega} \times \mathbb{R})$. Then there holds*

$$|u(x) - u(x_0)| \le C|x - x_0| \quad \textit{for any } x \in \Omega \textit{ and } x_0 \in \partial\Omega$$

*where $C$ is a positive constant depending only on $\lambda, \Omega, |a_{ij}, b_i|_{L^\infty(\Omega)}, M = |u|_{L^\infty(\Omega)}$, $|f|_{L^\infty(\Omega\times[-M,M])}$ and $|\varphi|_{C^2(\bar{\Omega})}$ for some $\varphi \in C^2(\bar{\Omega})$ with $\varphi = u$ on $\partial\Omega$.*

*Proof.* For simplicity we assume $u = 0$ on $\partial\Omega$. As before set $L = a_{ij}D_{ij} + b_iD_i$. Then we have

$$L(\pm u) = \pm f \ge -F \quad \text{in } \Omega$$

where we denote $F = \sup_\Omega |f(\cdot, u)|$. Now fix $x_0 \in \partial\Omega$. We will construct a function $w$ such that

$$Lw \le -F \text{ in } \Omega, \; w(x_0) = 0, \; w|_{\partial\Omega} \ge 0.$$

Then by the maximum principle we have

$$-w \le u \le w \quad \text{in } \Omega.$$

Taking normal derivative at $x_0$, we have

$$\left|\frac{\partial u}{\partial n}(x_0)\right| \le \frac{\partial w}{\partial n}(x_0).$$

So we need to bound $\frac{\partial w}{\partial n}(x_0)$ independently of $x_0$.

Consider the exterior ball $B_R(y)$ with $\bar{B}_R(y) \cap \bar{\Omega} = \{x_0\}$. Define $d(x)$ as the distance from $x$ to $\partial B_R(y)$. Then we have

$$0 < d(x) < D \equiv \text{diam}(\Omega) \quad \text{for any } x \in \Omega.$$

In fact, $d(x) = |x - y| - R$ for any $x \in \Omega$. Consider $w = \psi(d)$ for some function $\psi$ defined in $[0, \infty)$. Then we need

$$\begin{aligned}
&\psi(0) = 0 && (\Longrightarrow w(x_0) = 0)\\
&\psi(d) > 0 \text{ for } d > 0 && (\Longrightarrow w|_{\partial\Omega} \ge 0)\\
&\psi'(0) \text{ is controlled .}
\end{aligned}$$

¿From the first two inequalities, it is natural to require that $\psi'(d) > 0$. Note

$$Lw = \psi'' a_{ij} D_i d D_j d + \psi' a_{ij} D_{ij} d + \psi' b_i D_i d.$$

Direct calculation yields

$$\begin{aligned}
D_i d(x) &= \frac{x_i - y_i}{|x - y|}\\
D_{ij} d(x) &= \frac{\delta_{ij}}{|x - y|} - \frac{(x_i - y_i)(x_i - y_i)}{|x - y|^3}
\end{aligned}$$

which imply $|Dd| = 1$ and with $\Lambda = sup|a_{ij}|$

$$a_{ij} D_{ij} d = \frac{a_{ii}}{|x - y|} - \frac{a_{ij}}{|x - y|} D_i d D_j d \le \frac{n\Lambda}{|x - y|} - \frac{\lambda}{|x - y|} = \frac{n\Lambda - \lambda}{|x - y|} \le \frac{n\Lambda - \lambda}{R}.$$

Therefore we obtain by ellipticity

$$\begin{aligned}
Lw &\le \psi'' a_{ij} D_i d D_j d + \psi' \left( \frac{n\Lambda - \lambda}{R} + |b| \right)\\
&\le \lambda \psi'' + \psi' \left( \frac{n\Lambda - \lambda}{R} + |b| \right)
\end{aligned}$$

if we require $\psi'' < 0$. Hence in order to have $Lw \le -F$ we need

$$\lambda\psi'' + \psi'\left(\frac{n\Lambda - \lambda}{R} + |b|\right) + F \le 0.$$

To this end, we study the equation for some positive constants $a$ and $b$

$$\psi'' + a\psi' + b = 0$$

whose solution is given by

$$\psi(d) = -\frac{b}{a}d + \frac{C_1}{a} - \frac{C_2}{a}e^{-ad}$$

for some constants $C_1$ and $C_2$. For $\psi(0) = 0$, we need $C_1 = C_2$. Hence we have for some constant $C$

$$\psi(d) = -\frac{b}{a}d + \frac{C}{a}(1 - e^{-ad})$$

which implies

$$\begin{aligned}\psi'(d) &= Ce^{-ad} - \frac{b}{a} = e^{-ad}\left(C - \frac{b}{a}e^{ad}\right)\\ \psi''(d) &= -Cae^{-ad}.\end{aligned}$$

In order to have $\psi'(d) > 0$, we need $C \ge \frac{b}{a}e^{aD}$. Since $\psi'(d) > 0$ for $d > 0$, so $\psi(d) > \psi(0) = 0$ for any $d > 0$. Therefore we take

$$\begin{aligned}\psi(d) &= -\frac{b}{a}d + \frac{b}{a^2}e^{aD}(1 - e^{-ad})\\ &= \frac{b}{a}\left\{\frac{1}{a}e^{aD}(1 - e^{-ad}) - d\right\}.\end{aligned}$$

Such $\psi$ satisfies all the requirements we imposed. This finishes the proof.

## §4. Alexandroff Maximum Principle

Suppose $\Omega$ is a bounded domain in $\mathbb{R}^n$ and consider a second order elliptic operator $L$ in $\Omega$

$$L \equiv a_{ij}(x)D_{ij} + b_i(x)D_i + c(x)$$

where coefficients $a_{ij}, b_i, c$ are at least continuous in $\Omega$. Ellipticity means that the coefficient matrix $A = (a_{ij})$ is positive definite everywhere in $\Omega$. We set $D = det(A)$

and $D^* = D^{\frac{1}{n}}$ so that $D^*$ is the geometric mean of the eigenvalues of $A$. Throughout this section we assume

$$0 < \lambda \le D^* \le \Lambda$$

where $\lambda$ and $\Lambda$ are two positive constants, which denote, respectively, the minimal and maximal eigenvalues of $A$.

Before stating the main theorem we first introduce the concept of contact sets. For $u \in C^2(\Omega)$ we define

$$\Gamma^+ = \{y \in \Omega;\ u(x) \le u(y) + Du(y) \cdot (x - y),\ \text{for any } x \in \Omega\}.$$

The set $\Gamma^+$ is called the upper contact set of $u$ and Hessian matrix $D^2u = (D_{ij}u)$ is nonpositive on $\Gamma^+$. In fact upper contact set can also be defined for continuous function $u$ by the following

$$\Gamma^+ = \{y \in \Omega;\ u(x) \le u(y) + p \cdot (x - y),\ \text{for any } x \in \Omega \ \text{ and some } \ p = p(y) \in \mathbb{R}^n\}.$$

Clearly, $u$ is concave if and only if $\Gamma^+ = \Omega$. If $u \in C^1(\Omega)$, then $p(y) = Du(y)$ and any support hyperplane must then be a tangent plane to the graph.

Now we consider the equation of the following form

$$Lu = f \quad \text{in } \Omega$$

for some $f \in C(\Omega)$.

**Theorem 4.1.** *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies $Lu \ge f$ in $\Omega$ with the following conditions*

$$\frac{|b|}{D^*}, \frac{f}{D^*} \in L^n(\Omega), \quad \text{and} \quad c \le 0 \ \text{ in } \ \Omega.$$

*Then there holds*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C \| \frac{f^-}{D^*} \|_{L^n(\Gamma^+)}$$

*where $\Gamma^+$ is the upper contact set of $u$ and $C$ is a constant depending only on $n$, diam$(\Omega)$ and $\| \frac{b}{D^*} \|_{L^n(\Gamma^+)}$. In fact, $C$ can be written as*

$$d \cdot \left\{ exp\{ \frac{2^{n-2}}{\omega_n n^n} \big( \| \frac{b}{D^*} \|^n_{L^n(\Gamma^+)} + 1 \big) \} - 1 \right\}$$

*with $\omega_n$ as the volume of the unit ball in $\mathbb{R}^n$.*

*Remark.* The integral domain $\Gamma^+$ can be replaced by

$$\Gamma^+ \cap \{x \in \Omega; u(x) > \sup_{\partial\Omega} u^+\}.$$

*Remark.* There is no assumption on uniform ellipticity. Compare with the Hopf's maximum principle in Section 1.

We need a lemma first.

**Lemma 4.2.** *Suppose $g \in L^1_{loc}(\mathbb{R}^n)$ is nonnegative. Then for any $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ there holds*

$$\int_{B_{\tilde{M}}(0)} g \leq \int_{\Gamma^+} g(Du)|detD^2u|$$

*where $\Gamma^+$ is the upper contact set of $u$ and $\tilde{M} = (\sup_{\Omega} u - \sup_{\partial\Omega} u^+)/d$ with $d = diam(\Omega)$.*

*Remark.* For any positive definite matrix $A = (a_{ij})$ we have

$$det(-D^2u) \leq \frac{1}{D}\left(\frac{-a_{ij}D_{ij}u}{n}\right)^n \quad \text{on } \Gamma^+.$$

Hence we have another form for Lemma 4.2

$$\int_{B_{\tilde{M}}(0)} g \leq \int_{\Gamma^+} g(Du)\big(\frac{-a_{ij}D_{ij}u}{nD^*}\big)^n.$$

*Remark.* A special case corresponds to $g = 1$:

$$\begin{aligned}\sup_{\Omega} u &\leq \sup_{\partial\Omega} u^+ + \frac{d}{\omega_n^{\frac{1}{n}}}\left(\int_{\Gamma^+} |detD^2u|\right)^{\frac{1}{n}} \\ &\leq \sup_{\partial\Omega} u^+ + \frac{d}{\omega_n^{\frac{1}{n}}}\left(\int_{\Gamma^+}\left(-\frac{a_{ij}D_{ij}u}{nD^*}\right)^n\right)^{\frac{1}{n}}.\end{aligned}$$

Note this is the Theorem 4.1 if $b_i \equiv 0$ and $c \equiv 0$.

*Proof of Lemma 4.2.* Without loss of generality we assume $u \leq 0$ on $\partial\Omega$. Set $\Omega^+ = \{u > 0\}$. By the area-formula for $Du$ in $\Gamma^+ \cap \Omega^+ \subset \Omega$, we have

$$\int_{Du(\Gamma^+\cap\Omega^+)} g \leq \int_{\Gamma^+\cap\Omega^+} g(Du)|det(D^2u)|, \tag{1}$$

where $|det(D^2u)|$ is the Jacobian of the map $Du : \Omega \to \mathbb{R}^n$. In fact we may consider $\chi_\varepsilon = Du - \varepsilon\text{Id} : \Omega \to \mathbb{R}^n$. Then $D\chi_\varepsilon = D^2u - \varepsilon I$, which is negative definite in $\Gamma^+$. Hence by change of variable formula we have

$$\int_{\chi_\varepsilon(\Gamma^+\cap\Omega^+)} g = \int_{\Gamma^+\cap\Omega^+} g(\chi_\varepsilon)|det(D^2u - \varepsilon I)|,$$

which implies (1) if we let $\varepsilon \to 0$.

Now we claim $B_{\tilde{M}}(0) \subset Du(\Gamma^+ \cap \Omega^+)$, i.e., for any $a \in \mathbb{R}^n$ with $|a| < \tilde{M}$ there exists $x \in \Gamma^+ \cap \Omega^+$ such that $a = Du(x)$.

We may assume $u$ attains its maximum $m > 0$ at $0 \in \Omega$, i.e.,

$$u(0) = m = \sup_{\Omega} u.$$

Consider an affine function for $|a| < m/d (\equiv \tilde{M})$

$$L(x) = m + a \cdot x.$$

Then $L(x) > 0$ for any $x \in \Omega$ and $L(0) = m$. Since $u$ assumes its maximum at 0, then $Du(0) = 0$. Hence there exists an $x_1$ close to 0 such that $u(x_1) > L(x_1) > 0$. Note that $u \leq 0 < L$ on $\partial\Omega$. Hence there exists an $\tilde{x} \in \Omega$ such that $Du(\tilde{x}) = DL(\tilde{x}) = a$. Now we may translate vertically the plane $y = L(x)$ to the highest such position, i,e., the whole surface $y = u(x)$ lies below the plane. Clearly at such point, the function $u$ is positive.

*Proof of Theorem 4.1.* We should choose $g$ appropriately in order to apply Lemma 4.2. Note if $f \equiv 0$ and $c \equiv 0$ then $(-a_{ij}D_{ij}u)^n \leq |b|^n |Du|^n$ in $\Omega$. This suggests that we should take $g(p) = |p|^{-n}$. However such function is not locally integrable (at origin). Hence we will choose $g(p) = (|p|^n + \mu^n)^{-1}$ and then let $\mu \to 0^+$.

First we have by Cauchy inequality

$$\begin{aligned} -a_{ij}D_{ij} &\leq b_i D_i u + cu - f \\ &\leq b_i D_i u - f \quad \text{ in } \Omega^+ = \{x; u(x) > 0\} \\ &\leq |b| \cdot |Du| + f^- \\ &\leq \left( |b|^n + \frac{(f^-)^n}{\mu^n} \right)^{\frac{1}{n}} \cdot (|Du|^n + \mu^n)^{\frac{1}{n}} \cdot (1+1)^{\frac{n-2}{n}}, \end{aligned}$$

in particular

$$(-a_{ij}D_{ij}u)^n \leq \left( |b|^n + (\frac{f^-}{\mu})^n \right) (|Du|^n + \mu^n) \cdot 2^{n-2}.$$

Now we choose

$$g(p) = \frac{1}{|p|^n + \mu^n}.$$

By Lemma 4.2 we have

$$\int_{B_{\tilde{M}}(0)} g \leq \frac{2^{n-2}}{n^n} \int_{\Gamma^+ \cap \Omega^+} \frac{|b^n| + \mu^{-n}(f^-)^n}{D}.$$

We evaluate the integral in the left hand side in the following way

$$\int\limits_{B_{\tilde{M}}(0)} g = \omega_n \int_0^{\tilde{M}} \frac{r^{n-1}}{r^n + \mu^n} dr = \frac{\omega_n}{n} \log \frac{\tilde{M}^n + \mu^n}{\mu^n} = \frac{\omega_n}{n} \log \left(\frac{\tilde{M}^n}{\mu^n} + 1\right).$$

Therefore we obtain

$$\tilde{M}^n \leq \mu^n \left\{ exp \left\{ \frac{2^{n-2}}{\omega_n n^n} \left[ \| \frac{b}{D^*} \|^n_{L^n(\Gamma^+ \cap \Omega^+)} + \mu^{-n} \| \frac{f^-}{D^*} \|^n_{L^n(\Gamma^+ \cap \Omega^+)} \right] \right\} - 1 \right\}.$$

If $f \not\equiv 0$, we choose $\mu = \| \frac{f^-}{D^*} \|_{L^n(\Gamma^+ \cap \Omega^+)}$. If $f \not\equiv 0$, we may choose any $\mu > 0$ and then let $\mu \to 0$.

In the following we use Theorem 4.1 and Lemma 4.2 to derive some a-priori estimates for solutions to quasilinear equations and fully nonlinear equations. In the next result we do not assume uniform ellipticity.

**Proposition 4.3.** *Suppose that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$Qu \equiv a_{ij}(x, u, Du) D_{ij} u + b(x, u, Du) = 0 \quad \text{in } \Omega$$

*where $a_{ij} \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfies*

$$a_{ij}(x, z, p) \xi_i \xi_j > 0 \text{ for any } (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^n.$$

*Suppose there exist non-negative functions $g \in L^n_{loc}(\mathbb{R}^n)$ and $h \in L^n(\Omega)$ such that*

$$\frac{|b(x, z, p)|}{nD^*} \leq \frac{h(x)}{g(p)} \text{ for any } (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$$

*and*

$$\int\limits_{\Omega} h^n(x) dx < \int\limits_{\mathbb{R}^n} g^n(p) dp \equiv g_\infty.$$

*Then there holds*

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C\, diam(\Omega),$$

*where $C$ is a positive constant depending only on $g$ and $h$.*

*Example.* The prescribed mean curvature equation is given by

$$(1 + |Du|^2) \triangle u - D_i u D_j u D_{ij} u = nH(x)(1 + |Du|^2)^{\frac{3}{2}}$$

for some $H \in C(\Omega)$. We have

$$a_{ij}(x,z,p) = (1+|p|^2)\delta_{ij} - p_i p_j \Rightarrow D = (1+|p|^2)^{n-1}$$
$$b = -nH(x)(1+|p|^2)^{\frac{3}{2}}.$$

This implies

$$\frac{|b(x,z,p)|}{nD^*} \leq \frac{|H(x)|(1+|p|^2)^{\frac{3}{2}}}{(1+|p|^2)^{\frac{n-1}{n}}} = |H(x)|(1+|p|^2)^{\frac{n+2}{2n}}$$

and in particular

$$g_\infty = \int_{\mathbb{R}^n} g^n(p)dp = \int_{\mathbb{R}^n} \frac{dp}{(1+|p|^2)^{\frac{n+2}{2}}} = \omega_n.$$

**Corollary 4.4.** *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$(1+|Du|^2)\triangle u - D_i u D_j u D_{ij} u = nH(x)(1+|Du|^2)^{\frac{3}{2}} \text{ in } \Omega$$

*for some $H \in C(\Omega)$. Then if*

$$H_0 \equiv \int_\Omega |H(x)|^n dx < \omega_n$$

*we have*

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C\,diam(\Omega)$$

*where $C$ is a positive constant depending only on $n$ and $H_0$.*

*Proof of Proposition 4.3.* We prove for subsolutions. Assume $Qu \geq 0$ in $\Omega$. Then we have

$$-a_{ij} D_{ij} u \leq b \quad \text{in } \Omega.$$

Note that $\{D_{ij}u\}$ is nonpositive in $\Gamma^+$. Hence $-a_{ij}D_{ij}u \geq 0$, which implies $b(x,u,Du) \geq 0$ in $\Gamma^+$. Then in $\Gamma^+ \cap \Omega^+$ there holds

$$\frac{b(x,z,Du)}{nD^*} \leq \frac{h(x)}{g(Du)}.$$

We may apply Lemma 4.2 to $g^n$ and get

$$\int_{B_{\tilde{M}}(0)} g^n \leq \int_{\Gamma^+\cap\Omega^+} g^n(Du)\Big(\frac{-a_{ij}D_{ij}u}{nD^*}\Big)^n \leq \int_{\Gamma^+\cap\Omega^+} g^n(Du)\Big(\frac{b}{nD^*}\Big)^n$$
$$\leq \int_{\Gamma^+\cap\Omega^+} h^n \leq \int_\Omega h^n \Big(< \int_{\mathbb{R}^n} g^n\Big).$$

Therefore there exists a positive constant $C$, depending only on $g$ and $h$, such that $\tilde{M} \leq C$. This implies

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \text{ diam}(\Omega).$$

Next we discuss Monge-Ampére equations.

**Proposition 4.5.** *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$det(D^2u) = f(x, u, Du) \quad in\ \Omega$$

*for some $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Suppose there exist nonnegative functions $g \in L^1_{loc}(\mathbb{R}^n)$ and $h \in L^1(\Omega)$ such that*

$$|f(x, z, p)| \leq \frac{h(x)}{g(p)} \quad for\ any\ (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$$

*and*

$$\int_{\Omega} h(x)dx < \int_{\mathbb{R}^n} g(p)dp \equiv g_\infty.$$

*Then there holds*

$$\sup_{\Omega}|u| \leq \sup_{\partial\Omega}|u| + C\ diam(\Omega),$$

*where $C$ is a positive constant depending only on $g$ and $h$.*

The proof is similar to that of Proposition 4.3. There are two special cases. The first case is given by $f = f(x)$. We may take $g \equiv 1$ and hence $g_\infty = \infty$. So we obtain

**Corollary 4.6.** *Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy*

$$det(D^2u) = f(x) \quad in\ \Omega$$

*for some $f \in C(\bar{\Omega})$. Then there holds*

$$\sup_{\Omega}|u| \leq \sup_{\partial\Omega}|u| + \frac{diam(\Omega)}{\omega_n^{\frac{1}{n}}} \left( \int_{\Omega} |f|^n \right)^{\frac{1}{n}}.$$

Second case is about the prescribed Gaussian curvature equations.

**Corollary 4.7.** *Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy*

$$det(D^2u) = K(x)(1 + |Du|^2)^{\frac{n+2}{2}} \quad in\ \Omega$$

*for some $K \in C(\bar{\Omega})$. Then if*

$$K_0 \equiv \int_\Omega |K(x)| < \omega_n$$

*we have*

$$\sup_\Omega |u| \le \sup_{\partial\Omega} |u| + C\,diam(\Omega)$$

*where $C$ is a positive constant depending only on $n$ and $K_0$.*

We finish this section by proving a maximum principle in a domain with small volume, which is due to Varadham.

Consider

$$Lu \equiv a_{ij}D_{ij}u + b_iD_iu + cu \quad \text{in} \ \ \Omega$$

where $\{a_{ij}\}$ is positive definitely pointwisely in $\Omega$ and

$$|b_i| + |c| \le \Lambda \quad \text{and } \det(a_{ij}) \ge \lambda$$

for some positive constants $\lambda$ and $\Lambda$.

**Theorem 4.8.** *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies $Lu \ge 0$ in $\Omega$ with $u \le 0$ on $\partial\Omega$. Assume $diam(\Omega) \le d$. Then there is a positive constant $\delta = \delta(n, \lambda, \Lambda, d) > 0$ such that if $|\Omega| \le \delta$ then $u \le 0$ in $\Omega$.*

*Proof.* If $c \le 0$, then $u \le 0$ by Theorem 4.1. In general write $c = c^+ - c^-$. Then

$$a_{ij}D_{ij}u + b_iD_iu - c^-u \ge -c^+u (\equiv f).$$

By Theorem 4.1 we have

$$\begin{aligned}\sup_\Omega u &\le c(n, \lambda, \Lambda, d)\|c^+u^+\|_{L^n(\Omega)} \\ &\le c(n, \lambda, \Lambda, d)\|c^+\|_{L^\infty}|\Omega|^{\frac{1}{n}} \cdot \sup_\Omega u \le \frac{1}{2}\sup_\Omega u\end{aligned}$$

if $|\Omega|$ is small. Hence we get $u \le 0$ in $\Omega$.

*Remark.* Compare this with Proposition 1.9, the maximum principle for narrow domain.

## §5. Moving Plane Method

In this section we will use the moving plane method to discuss the symmetry of solutions. The following result was first proved by Gidas, Ni and Nirenberg.

**Theorem 5.1.** *Suppose $u \in C(\bar{B}_1) \cap C^2(B_1)$ is a positive solution of*

$$\begin{aligned} \Delta u + f(u) = 0 \ &in\ B_1 \\ u = 0 \ &on\ \partial B_1 \end{aligned}$$

*where $f$ is locally Lipschitz in $\mathbb{R}$. Then $u$ is radially symmetric in $B_1$ and $\frac{\partial u}{\partial r}(x) < 0$ for $x \neq 0$.*

The original proof requires that solutions be $C^2$ up to the boundary. Here we give a method which does not depend on the smoothness of domains nor the smoothness of solutions up to the boundary.

**Lemma 5.2.** *Suppose that $\Omega$ is a bounded domain which is convex in $x_1$ direction and symmetric with respect to the plane $\{x_1 = 0\}$. Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ is a positive solution of*

$$\begin{aligned} \Delta u + f(u) = 0 \ &in\ \Omega \\ u = 0 \ &on\ \partial \Omega \end{aligned}$$

*where $f$ is locally Lipschitz in $\mathbb{R}$. Then $u$ is symmetric with respect to $x_1$ and $D_{x_1} u(x) < 0$ for any $x \in \Omega$ with $x_1 > 0$.*

*Proof.* Write $x = (x_1, y) \in \Omega$ for $y \in \mathbb{R}^{n-1}$. We will prove

(1) $$u(x_1, y) < u(x_1^*, y) \text{ for any } x_1 > 0 \text{ and } x_1^* < x_1 \ \text{ with } x_1^* + x_1 > 0.$$

Then by letting $x_1^* \to -x_1$, we get $u(x_1, y) \leq u(-x_1, y)$ for any $x_1$. Then by changing the direction $x_1 \to -x_1$, we get the symmetry.

Let $a = \sup x_1$ for $(x_1, y) \in \Omega$. For $0 < \lambda < a$, define

$$\begin{aligned} \Sigma_\lambda &= \{x \in \Omega; x_1 > \lambda\} \\ T_\lambda &= \{x_1 = \lambda\} \\ \Sigma'_\lambda &= \text{the reflection of } \Sigma_\lambda \text{ with respect to } T_\lambda \\ x_\lambda &= (2\lambda - x_1, x_2, \cdots x_n) \text{ for } x = (x_1, x_2, \cdots x_n). \end{aligned}$$

In $\Sigma_\lambda$ we define

$$w_\lambda(x) = u(x) - u(x_\lambda) \text{ for } x \in \Sigma_\lambda.$$

Then we have by Mean Value Theorem

$$\begin{aligned} \Delta w_\lambda + c(x, \lambda) w_\lambda &= 0 \text{ in } \Sigma_\lambda \\ w_\lambda \leq 0 \text{ and } w_\lambda &\not\equiv 0 \text{ on } \partial \Sigma_\lambda. \end{aligned}$$

where $c(x, \lambda)$ is a bounded function in $\Sigma_\lambda$.

We need to show $w_\lambda < 0$ in $\Sigma_\lambda$ for any $\lambda \in (0, a)$. This implies in particular that $w_\lambda$ assumes along $\partial\Sigma_\lambda \cap \Omega$ its maximum in $\Sigma_\lambda$. By Theorem 1.3 (Hopf Lemma) we have for any such $\lambda \in (0, a)$

$$D_{x_1} w_\lambda \Big|_{x_1=\lambda} = 2 D_{x_1} u \Big|_{x_1=\lambda} < 0.$$

For any $\lambda$ close to $a$, we have $w_\lambda < 0$ by Proposition 1.9 (the maximum principle for narrow domain) or Theorem 4.8. Let $(\lambda_0, a)$ be the largest interval of values of $\lambda$ such that $w_\lambda < 0$ in $\Sigma_\lambda$. We want to show $\lambda_0 = 0$. If $\lambda_0 > 0$, by continuity, $w_{\lambda_0} \le 0$ in $\Sigma_{\lambda_0}$ and $w_{\lambda_0} \not\equiv 0$ on $\partial\Sigma_{\lambda_0}$. Then Theorem 1.4 (strong maximum principle) implies $w_{\lambda_0} < 0$ in $\Sigma_{\lambda_0}$. We will show that for any small $\varepsilon > 0$

$$w_{\lambda_0-\varepsilon} < 0 \text{ in } \Sigma_{\lambda_0-\varepsilon}.$$

Fix $\delta > 0$ (to be determined). Let $K$ be a closed subset in $\Sigma_{\lambda_0}$ such that $|\Sigma_{\lambda_0} \setminus K| < \delta/2$. The fact $w_{\lambda_0} < 0$ in $\Sigma_{\lambda_0}$ implies

$$w_{\lambda_0}(x) \le -\eta < 0 \text{ for any } x \in K.$$

By continuity we have

$$w_{\lambda_0-\varepsilon} < 0 \text{ in } K.$$

For $\varepsilon > 0$ small, $|\Sigma_{\lambda_0-\varepsilon} \setminus K| < \delta$. We choose $\delta$ in such a way that we may apply Theorem 4.8 (maximum principle for domain with small volume) to $w_{\lambda_0-\varepsilon}$ in $\Sigma_{\lambda_0-\varepsilon} \setminus K$. Hence we get

$$w_{\lambda_0-\varepsilon}(x) \le 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \setminus K$$

and then by Theorem 1.7

$$w_{\lambda_0-\varepsilon}(x) < 0 \text{ in } \Sigma_{\lambda_0-\varepsilon} \setminus K.$$

Therefore we obtain for any small $\varepsilon > 0$

$$w_{\lambda_0-\varepsilon}(x) < 0 \text{ in } \Sigma_{\lambda_0-\varepsilon}.$$

This contradicts the choice of $\lambda_0$.

# CHAPTER 3
# WEAK SOLUTIONS, PART I
## GUIDE

The first section provide some general knowledge of Campanato and BMO spaces that are needed in both Chapter 3 and 4.Sections 3.2,3.3 and sections 5.3,5.4 can be viewed as perturbation theory (from constant cofficients equations).The former deals with equations of divergence type,and the latter is for nondivergence type equations. The classical theory of Schauder estimates and $L^p$ estimates are also contained in the latter treatment.Note we did not use the classical potential estimates.Here two papers by Caffarelli [C1,2] and the book of Giaquinta [G] are sources for the further readings.

In this chapter and the next we discuss various regularity results for weak solutions to elliptic equations of divergence form. In order to explain ideas clearly we will discuss the equations with the following form only

$$-D_j(a_{ij}(x)D_iu) + c(x)u = f(x).$$

We assume that $\Omega$ is a domain in $\mathbb{R}^n$. The function $u \in H^1(\Omega)$ is a *weak solution* if it satisfies

$$\int_\Omega \big(a_{ij}D_iuD_j\varphi + cu\varphi\big) = \int_\Omega f\varphi \quad \text{for any } \varphi \in H_0^1(\Omega),$$

where we assume

(i) the leading coefficients $a_{ij} \in L^\infty(\Omega)$ are *uniformly elliptic*, i.e., for some positive constant $\lambda$ there holds

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n;$$

(ii) the coefficient $c \in L^{\frac{n}{2}}(\Omega)$ and nonhomogenuous term $f \in L^{\frac{2n}{n+2}}(\Omega)$.

Note by Sobolev embedding theorem (ii) is the least assumption on $c$ and $f$ to have a meaningful equation.

We will prove various interior regularity results concerning the solution $u$ if we have better assumptions on coefficients $a_{ij}$ and $c$ and nonhomogenuous term $f$. Basically there are two class of regularity results, perturbation results and nonperturbation results. The first is based on the regularity assumption on the leading coefficients $a_{ij}$, which are assumed to be at least continuous. Under such assumption we may compare solutions to the underlying equations with harmonic functions, or solutions to constant coefficient equations. Then the regularity of solutions depends on how close they are to harmonic functions or how close the leading coefficients $a_{ij}$ are to constant coefficients.

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In this direction we have Schauder estimates and $W^{2,p}$ estimates. In this chapter we only discuss the Schauder estimates. For the second kind of regularity, there is no continuity assumption on the leading coefficients $a_{ij}$. Hence the result is not based on the perturbation. The iteration methods introduced by DeGiorgi and Moser are successful in dealing with nonperturbation situation. The results proved by them are fundamental for the discussion of quasilinear equations, where the coefficients depend on the solutions. In fact the linearlity has no bearing in their arguments. This permits an extension of these results to quasilinear equations with appropriate structure conditions.

One may discuss boundary regularities in a similar way. We leave the details to readers.

## §1. Growth of Local Integrals

Let $B_R(x_0)$ be the ball in $\mathbb{R}^n$ of radius $R$ centered at $x_0$. The well-known Sobolev theorem states that if $u \in W^{1,p}(B_R(x_0))$ with $p > n$ then $u$ is Hölder continuous with exponent $\alpha = 1 - n/p$.

In the first part of this section we prove a general result, due to S. Campanato, which characterizes Hölder continuous functions by the growth of their local integrals. This result will be very useful for studying the regularity of solutions to elliptic differential equations. In the second part of this section we prove a result, due to John and Nirenberg, which gives an equivalent definition of functions of bounded mean oscillation.

Let $\Omega$ be a bounded connected open set in $\mathbb{R}^n$ and let $u \in L^1(\Omega)$. For any ball $B_r(x_0) \subset \Omega$, define

$$u_{x_0,r} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u.$$

**Theorem 1.1.** *Suppose $u \in L^2(\Omega)$ satisfies*

$$\int_{B_r(x)} |u - u_{x,r}|^2 \leq M^2 r^{n+2\alpha}, \text{ for any } B_r(x) \subset \Omega,$$

*for some $\alpha \in (0,1)$. Then $u \in C^\alpha(\Omega)$ and for any $\Omega' \subset\subset \Omega$ there holds*

$$\sup_{\Omega'} |u| + \sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq c\big\{M + \|u\|_{L^2(\Omega)}\big\}$$

*where $c = c(n, \alpha, \Omega, \Omega') > 0$.*

*Proof.* Denote $R_0 = dist(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$ and $0 < r_1 < r_2 \leq R_0$, we have

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leq 2(|u(x) - u_{x_0,r_1}|^2 + |u(x) - u_{x_0,r_2}|^2)$$

and integrating with respect to $x$ in $B_{r_1}(x_0)$

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leq \frac{2}{\omega_n r_1^n}\{\int_{B_{r_1}(x_0)} |u - u_{x_0,r_1}|^2 + \int_{B_{r_2}(x_0)} |u - u_{x_0,r_2}|^2\}$$

from which the estimate

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leq c(n) M^2 r_1^{-n}\{r_1^{n+2\alpha} + r_2^{n+2\alpha}\}, \tag{1}$$

follows.

For any $R \leq R_0$, with $r_1 = R/2^{i+1}, r_2 = R/2^i$, we obtain

$$|u_{x_0,2^{-(i+1)}R} - u_{x_0,2^{-i}R}| \leq c(n) 2^{-(i+1)\alpha} M R^\alpha$$

and therefore for $h < k$

$$|u_{x_0,2^{-h}R} - u_{x_0,2^{-k}R}| \leq \frac{c(n)}{2^{(h+1)\alpha}} M R^\alpha \sum_{i=0}^{k-h-1} \frac{1}{2^{i\alpha}} \leq \frac{c(n,\alpha)}{2^{h\alpha}} M R^\alpha.$$

This shows that $\{u_{x_0,2^{-i}R}\} \subset \mathbb{R}$ is a Cauchy sequence, hence a convergent one. Its limit $\hat{u}(x_0)$ is independent of the choice of $R$, since (1) can be applied with $r_1 = 2^{-i}R$ and $r_2 = 2^{-i}\bar{R}$ whenever $0 < R < \bar{R} \leq R_0$. Thus we get

$$\hat{u}(x_0) = \lim_{r \downarrow 0} u_{x_0,r}$$

with

$$|u_{x_0,r} - \hat{u}(x_0)| \leq c(n,\alpha) M r^\alpha \tag{2}$$

for any $0 < r \leq R_0$.

Recall that $\{u_{x,r}\}$ converges, as $r \to 0+$, in $L^1(\Omega)$ to the function $u$, by the Lebesgue theorem, so we have $u = \hat{u}$ a.e. and (2) implies that $\{u_{x,r}\}$ converges uniformly to $u(x)$ in $\Omega'$. Since $x \mapsto u_{x,r}$ is continuous for any $r > 0$, $u(x)$ is continuous. By (2) we get

$$|u(x)| \leq CMR^\alpha + |u_{x,R}|$$

for any $x \in \Omega'$ and $R \leq R_0$. Hence $u$ is bounded in $\Omega'$ with the estimate

$$\sup_{\Omega'} |u| \leq c\{M R_0^\alpha + \|u\|_{L^2(\Omega)}\}.$$

Finally we prove that $u$ is Hölder continuous. Let $x, y \in \Omega'$ with $R = |x-y| < R_0/2$. Then we have

$$|u(x) - u(y)| \le |u(x) - u_{x,2R}| + |u(y) - u_{y,2R}| + |u_{x,2R} - u_{y,2R}|.$$

The first two terms on the right sides are estimated in (2). For the last term we write

$$|u_{x,2R} - u_{y,2R}| \le |u_{x,2R} - u(\zeta)| + |u_{y,2R} - u(\zeta)|$$

and integrating with respect to $\zeta$ over $B_{2R}(x) \cap B_{2R}(y)$, which contains $B_R(x)$, yields

$$\begin{aligned} |u_{x,2R} - u_{y,2R}|^2 &\le \frac{2}{|B_R(x)|}\{\int_{B_{2R}(x)} |u - u_{x,2R}|^2 + \int_{B_{2R}(y)} |u - u_{y,2R}|^2\} \\ &\le c(n,\alpha) M^2 R^{2\alpha}. \end{aligned}$$

Therefore we have

$$|u(x) - u(y)| \le c(n,\alpha) M |x-y|^\alpha.$$

For $|x-y| > R_0/2$ we obtain

$$|u(x) - u(y)| \le 2 \sup_{\Omega'} |u| \le c\{M + \frac{1}{R_0^\alpha}\|u\|_{L^2}\}|x-y|^\alpha.$$

This finishes the proof.

The Sobolev theorem is an easy consequence of Theorem 1.1. In fact we have the following result due to Morrey.

**Corollary 1.2.** *Suppose $u \in H^1_{loc}(\Omega)$ satisfies*

$$\int_{B_r(x)} |Du|^2 \le M^2 r^{n-2+2\alpha}, \text{ for any } B_r(x) \subset \Omega,$$

*for some $\alpha \in (0,1)$. Then $u \in C^\alpha(\Omega)$ and for any $\Omega' \subset\subset \Omega$ there holds*

$$\sup_{\Omega'} |u| + \sup_{\substack{x,y \in \Omega' \\ x \ne y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \le c\{M + \|u\|_{L^2(\Omega)}\}$$

*where $c = c(n, \alpha, \Omega, \Omega') > 0$.*

*Proof.* By Poincaré inequality, we obtain

$$\int_{B_r(x)} |u - u_{x,r}|^2 \le c(n) r^2 \int_{B_r(x)} |Du|^2 \le c(n) M^2 r^{n+2\alpha}.$$

By applying Theorem 1.1, we have the result.

The following result will be needed in section 2.

**Lemma 1.3.** *Suppose $u \in H^1(\Omega)$ satisfies*

$$\int_{B_r(x_0)} |Du|^2 \le M r^\mu, \text{ for any } B_r(x_0) \subset \Omega,$$

*for some $\mu \in [0, n)$. Then for any $\Omega' \subset\subset \Omega$ there holds for any $B_r(x_0) \subset \Omega$ with $x_0 \in \Omega'$*

$$\int_{B_r(x_0)} |u|^2 \le c(n, \lambda, \mu, \Omega, \Omega')\{M + \int_\Omega u^2\} r^\lambda$$

*where $\lambda = \mu + 2$ if $\mu < n - 2$ and $\lambda$ is any number in $[0, n)$ if $n - 2 \le \mu < n$.*

*Proof.* As before denote $R_0 = dist(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$ and $0 < r \le R_0$, Poincaré inequality yields

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \le c r^2 \int_{B_r(x_0)} |Du|^2 dx \le c(n) M r^{\mu+2}.$$

This implies that

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \le c(n) M r^\lambda$$

where $\lambda$ is as in the Theorem 1.3. For any $0 < \rho < r \le R_0$ we have

$$\begin{aligned}
\int_{B_\rho(x_0)} u^2 &\le 2 \int_{B_\rho(x_0)} |u_{x_0,r}|^2 + 2 \int_{B_\rho(x_0)} |u - u_{x_0,r}|^2 \\
&\le c(n) \rho^n |u_{x_0,r}|^2 + 2 \int_{B_r(x_0)} |u - u_{x_0,r}|^2 \\
&\le c(n) \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} u^2 + M r^\lambda
\end{aligned}$$

where we used

$$|u_{x_0,r}|^2 \le \frac{c(n)}{r^n} \int_{B_r(x_0)} u^2.$$

Hence the function $\phi(r) = \int_{B_r(x_0)} u^2$ satisfies the inequality

$$\phi(\rho) \le c(n)\{\left(\frac{\rho}{r}\right)^n \phi(r) + M r^\lambda\}, \text{ for any } 0 < \rho < r \le R_0 \tag{1}$$

for some $\lambda \in (0, n)$. If we may replace the term $Mr^\lambda$ in the right by $M\rho^\lambda$, we are done. In fact, we would obtain that for any $0 < \rho < r \le R_0$ there holds

$$\int_{B_\rho(x_0)} u^2 \le c\{\left(\frac{\rho}{r}\right)^\lambda \int_{B_r(x_0)} u^2 + M \rho^\lambda\}. \tag{2}$$

Choose $r = R_0$. This implies

$$\int_{B_\rho(x_0)} u^2 \le c \rho^\lambda \{\int_\Omega u^2 + M\} \text{ for any } \rho \le R_0.$$

In order to get (2) from (1), we need the following technical lemma.

**Lemma 1.4.** *Let $\phi(t)$ be a nonnegative and nondecreasing function on $[0, R]$. Suppose that*

$$\phi(\rho) \le A[\Big(\frac{\rho}{r}\Big)^{\alpha} + \varepsilon]\phi(r) + Br^{\beta}$$

*for any $0 < \rho \le r \le R$, with $A, B, \alpha, \beta$ nonnegative constants and $\beta < \alpha$. Then for any $\gamma \in (\beta, \alpha)$, there exists a constant $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta, \gamma)$ such that if $\varepsilon < \varepsilon_0$ we have for all $0 < \rho \le r \le R$*

$$\phi(\rho) \le c\{\Big(\frac{\rho}{r}\Big)^{\gamma}\phi(r) + B\rho^{\beta}\}$$

*where $c$ is a positive constant depending on $A, \alpha, \beta, \gamma$. In particular we have for any $0 < r \le R$*

$$\phi(r) \le c\{\frac{\phi(R)}{R^{\gamma}}r^{\gamma} + Br^{\beta}\}.$$

*Proof.* For $\tau \in (0, 1)$ and $r < R$, we have

$$\phi(\tau r) \le A\tau^{\alpha}[1 + \varepsilon\tau^{-\alpha}]\phi(r) + Br^{\beta}.$$

Choose $\tau < 1$ in such a way that $2A\tau^{\alpha} = \tau^{\gamma}$ and assume $\varepsilon_0\tau^{-\alpha} < 1$. Then we get for every $r < R$

$$\phi(\tau r) \le \tau^{\gamma}\phi(r) + Br^{\beta}$$

and therefore for all integers $k > 0$

$$\begin{aligned} \phi(\tau^{k+1}r) &\le \tau^{\gamma}\phi(\tau^{k}r) + B\tau^{k\beta}r^{\beta} \\ &\le \tau^{(k+1)\gamma}\phi(r) + B\tau^{k\beta}r^{\beta}\sum_{j=0}^{k}\tau^{j(\gamma-\beta)} \\ &\le \tau^{(k+1)\gamma}\phi(r) + \frac{B\tau^{k\beta}r^{\beta}}{1-\tau^{\gamma-\beta}}. \end{aligned}$$

Choosing $k$ such that $\tau^{k+2}r < \rho \le \tau^{k+1}r$, the last inequality gives

$$\phi(\rho) \le \frac{1}{\tau^{\gamma}}\Big(\frac{\rho}{r}\Big)^{\gamma}\phi(r) + \frac{B\rho^{\beta}}{\tau^{2\beta}(1-\tau^{\gamma-\beta})}.$$

In the rest of this section we discuss functions of bounded mean oscillation (BMO). The following result is proved by John and Nirenberg.

**Theorem 1.5 (John-Nirenberg Lemma).** *Suppose $u \in L^1(\Omega)$ satisfies*

$$\int_{B_r(x)} |u - u_{x,r}| \le Mr^n, \quad \textit{for any} \;\; B_r(x) \subset \Omega.$$

*Then there holds for any $B_r(x) \subset \Omega$*

$$\int_{B_r(x)} e^{\frac{p_0}{M}|u-u_{x,r}|} \le Cr^n,$$

*for some positive $p_0$ and $C$ depending only on $n$.*

*Remark.* Functions satisfying the condition of Theorem 1.5 are called functions of bounded mean oscillation ($BMO$). We have the following relation

$$L^\infty \underset{\neq}{\subset} BMO.$$

The counterexample is given by the following function in $(0,1) \subset \mathbb{R}$

$$u(x) = \log(x).$$

For convenience we use cubes instead of balls. We need the Calderon-Zygmund decomposition. First we introduce some terminology.

Take the unit cube $Q_0$. Cut it equally into $2^n$ cubes, which we take as the first generation. Do the same cutting for these small cubes to get the second generation. Continue this process. These cubes (from all generations) are called *dyadic cubes*. Any $(k+1)$-generation cube $Q$ comes from some $k$-generation cube $\tilde{Q}$, which is called the *predecessor* of $Q$.

**Lemma 1.6.** *Suppose $f \in L^1(Q_0)$ is nonnegative and $\alpha > |Q_0|^{-1} \int_{Q_0} f$ is a fixed constant. Then there exists a sequence of (nonoverlapping) dyadic cubes $\{Q_j\}$ in $Q_0$ such that*

$$f(x) \le \alpha \;\; \textit{a.e. in} \;\; Q_0 \setminus \cup_j Q_j$$

*and*

$$\alpha \le \frac{1}{|Q_j|} \int_{Q_j} f dx < 2^n \alpha.$$

*Proof.* Cut $Q_0$ into $2^n$ dyadic cubes and keep the cube $Q$ if $|Q|^{-1} \int_Q f \ge \alpha$. For others keep cutting and always keep the cube $Q$ if $|Q|^{-1} \int_Q f \ge \alpha$ and cut the rest. Let $\{Q_j\}$ be the cubes we have kept during this infinite process. We only need to verify that

$$f(x) \le \alpha \;\; \text{a.e. in} \;\; Q_0 \setminus \cup_j Q_j.$$

Let $F = Q_0 \setminus \cup_j Q_j$. For any $x \in F$, from the way we collect $\{Q_j\}$, there exists a sequence of cubes $Q^i$ containing $x$ such that

$$\frac{1}{|Q^i|}\int_{Q^i} f < \alpha$$

and

$$diam(Q^i) \to 0 \text{ as } i \to \infty.$$

By Lebesgue density theorem this implies that

$$f \leq \alpha \text{ a.e. in } F.$$

*Proof of Theorem 1.5.* Assume $\Omega = Q_0$. We may rewrite the assumption in terms of cubes as follows

$$\int_Q |u - u_Q| < M|Q|$$

for any $Q \subset Q_0$. We will prove that there exist two positive constants $c_1(n)$ and $c_2(n)$ such that for any $Q \subset Q_0$ there holds

$$|\{x \in Q; |u - u_Q| > t\}| \leq c_1|Q|exp\left(-\frac{c_2}{M}t\right).$$

Then Theorem 1.5 follows easily.

Assume without loss of generality $M = 1$. Choose $\alpha > 1 \geq |Q_0|^{-1}\int_{Q_0}|u - u_{Q_0}|dx$. Apply Calderon-Zygmund decomposition to $f = |u - u_{Q_0}|$. There exists a sequence of (nonoverlapping) cubes $\{Q_j^{(1)}\}_{j=1}^{\infty}$ such that

$$\alpha \leq \frac{1}{|Q_j^{(1)}|}\int_{Q_j^{(1)}} |u - u_{Q_0}| < 2^n\alpha$$

$$|u(x) - u_{Q_0}| \leq \alpha \;\; a.e.\; x \in Q_0 \setminus \cup_{j=1}^{\infty} Q_j^{(1)}.$$

This implies

$$\sum_j |Q_j^{(1)}| \leq \frac{1}{\alpha}\int_{Q_0} |u - u_{Q_0}| \leq \frac{1}{\alpha}|Q_0|$$

$$|u_{Q_j^{(1)}} - u_{Q_0}| \leq \frac{1}{|Q_j^{(1)}|}\int_{Q_j^{(1)}} |u - u_{Q_0}|dx \leq 2^n\alpha.$$

Definition of $BMO$ norm implies for each $j$

$$\frac{1}{|Q_j^{(1)}|}\int_{Q_j^{(1)}} |u - u_{Q_j{(1)}}|dx \leq 1 < \alpha.$$

Apply decomposition procedure above to $f = |u - u_{Q_j^{(1)}}|$ in $Q_j^{(1)}$. There exists a sequence of (nonoverlapping) cubes $\{Q_j^{(2)}\}$ in $\cup_j Q_j^{(1)}$ such that

$$\sum_{j=1}^{\infty} |Q_j^{(2)}| \leq \frac{1}{\alpha} \sum_j \int_{Q_j^{(1)}} |u - u_{Q_j^{(1)}}| \leq \frac{1}{\alpha} \sum_j |Q_j^{(1)}| \leq \frac{1}{\alpha^2} |Q_0|$$

and

$$|u(x) - u_{Q_j^{(1)}}| \leq \alpha \ \text{ a.e. } x \in Q_j^{(1)} \setminus \cup Q_j^{(2)},$$

which implies

$$|u(x) - u_{Q_0}| \leq 2 \cdot 2^n \alpha \text{ a.e. } x \in Q_0 \setminus \cup_j Q_j^{(2)}.$$

Continue this process. For any integer $k \geq 1$ there exists a sequence of disjoint cubes $\{Q_j^{(k)}\}$ such that

$$\sum_j |Q_j^{(k)}| \leq \frac{1}{\alpha^k} |Q_0|,$$

and

$$|u(x) - u_{Q_0}| \leq k 2^n \alpha \ a.e. \ x \in Q_0 \setminus \cup_j Q_j^{(k)}.$$

Thus

$$|\{x \in Q_0; |u - u_{Q_0}| > 2^n k \alpha\}| \leq \sum_{j=1}^{\infty} |Q_j^{(k)}| \leq \frac{1}{\alpha^k} |Q_0|.$$

For any $t$ there exists an integer $k$ such that $t \in [2^n k\alpha, 2^n(k+1)\alpha)$. This implies

$$\alpha^{-k} = \alpha \alpha^{-(k+1)} = \alpha e^{-(k+1)\log \alpha} \leq \alpha e^{-\frac{\log \alpha}{2^n \alpha} t}.$$

This finishes the proof.

## §2. **Hölder Continuity of Solutions**

In this section we will prove Hölder regularity for solutions. The basic idea is to freeze the leading coefficients and then to compare solutions with harmonic functions. The regularity of solutions depends on how close solutions are to harmonic functions. Hence we need some regularity assumption on the leading coefficients.

Suppose $a_{ij} \in L^\infty(B_1)$ is uniformly elliptic in $B_1 = B_1(0)$, i.e.,

$$\lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \text{ for any } x \in B_1, \xi \in \mathbb{R}^n.$$

In the following we assume that $a_{ij}$ is at least continuous. We assume that $u \in H^1(B_1)$ satisfies

$$(*) \qquad \int_{B_1} a_{ij} D_i u D_j \varphi + cu\varphi = \int_{B_1} f\varphi \quad \text{for any } \varphi \in H_0^1(B_1).$$

The main theorem we will prove are the following Hölder estimates for solutions.

**Theorem 2.1.** *Let $u \in H^1(B_1)$ solve $(*)$. Assume $a_{ij} \in C^0(\bar{B}_1)$, $c \in L^n(B_1)$ and $f \in L^q(B_1)$ for some $q \in (n/2,\ n)$. Then $u \in C^\alpha(B_1)$ with $\alpha = 2 - n/q \in (0,1)$. Moreover, there exists an $R_0 = R_0(\lambda, \Lambda, \tau, \|c\|_{L^n})$ such that for any $x \in B_{\frac{1}{2}}$ and $r \leq R_0$ there holds*

$$\int_{B_r(x)} |Du|^2 \leq Cr^{n-2+2\alpha}\big\{\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)}\big\}$$

*where $C = C(\lambda, \Lambda, \tau, \|c\|_{L^n})$ is a positive constant with*

$$|a_{ij}(x) - a_{ij}(y)| \leq \tau(|x-y|), \ \text{for any } x, y \in B_1.$$

*Remark.* In the case of $c \equiv 0$, we may replace $\|u\|_{H^1(B_1)}$ with $\|Du\|_{L^2(B_1)}$.

The idea of the proof is to compare the solution $u$ with harmonic functions and use the perturbation argument.

**Lemma 2.2.** *(Basic Estimates for Harmonic Functions.) Suppose $\{a_{ij}\}$ is a constant positive definite matrix with*

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n$$

*for some $0 < \lambda \leq \Lambda$. Suppose $w \in H^1(B_r(x_0))$ is a weak solution of*

$$(1) \qquad a_{ij} D_{ij} w = 0 \ \text{in } B_r(x_0).$$

*Then for any $0 < \rho \leq r$, there hold*

$$\int_{B_\rho(x_0)} |Dw|^2 \leq c\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Dw|^2$$

*and*

$$\int_{B_\rho(x_0)} |Dw - (Dw)_{x_0,\rho}|^2 \leq c\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Dw - (Dw)_{x_0,r}|^2$$

*where $c = c(\lambda, \Lambda)$.*

*Proof.* Note that if $w$ is a solution of (1) so is any of its derivatives. We may apply Lemma 4.4 in Chapter 1 to $Dw$.

**Corollary 2.3.** *(Comparison with Harmonic Functions.) Suppose $w$ is as in Lemma 2.2. Then for any $u \in H^1(B_r(x_0))$ there hold for any $0 < \rho \le r$*

$$\int_{B_\rho(x_0)} |Du|^2 \le c\Big\{ \Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} |Du|^2 + \int_{B_r(x_0)} |D(u-w)|^2 \Big\}$$

*and*

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le \Big\{ \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + \int_{B_r(x_0)} |D(u-w)|^2 \Big\}$$

*where $c$ is a positive constant depending only on $\lambda$ and $\Lambda$.*

*Proof.* We prove it by direct computation. In fact with $v = u - w$ we have for any $0 < \rho \le r$

$$\begin{aligned}\int_{B_\rho(x_0)} |Du|^2 &\le 2\int_{B_\rho(x_0)} |Dw|^2 + 2\int_{B_\rho(x_0)} |Dv|^2 \\ &\le c\Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} |Dw|^2 + 2\int_{B_r(x_0)} |Dv|^2 \\ &\le c\Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} |Du|^2 + c\Big[1 + \Big(\frac{\rho}{r}\Big)^n\Big] \int_{B_r(x_0)} |Dv|^2,\end{aligned}$$

and

$$\begin{aligned}\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 &\le 2\int_{B_\rho(x_0)} |Du - (Dw)_{x_0,\rho}|^2 + 2\int_{B_\rho(x_0)} |Dv|^2 \\ &\le 4\int_{B_\rho(x_0)} |Dw - (Dw)_{x_0,\rho}|^2 + 6\int_{B_\rho(x_0)} |Dv|^2 \\ &\le c\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Dw - (Dw)_{x_0,r}|^2 + 6\int_{B_r(x_0)} |Dv|^2 \\ &\le c\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + c\Big[1 + \Big(\frac{\rho}{r}\Big)^{n+2}\Big] \int_{B_r(x_0)} |Dv|^2.\end{aligned}$$

*Remark.* The regularity of $u$ depends on how close $u$ is to $w$, the solution to the constant coefficient equation.

We now prove the Theorem 2.1.

*Proof of Theorem 2.1.* We shall decompose $u$ into a sum $v + w$ where $w$ satisfies a homogeneous equation and $v$ has estimates in terms of nonhomogeneous terms.

For any $B_r(x_0) \subset B_1$ write the equation in the following form

$$\int_{B_1} a_{ij}(x_0) D_i u D_j \varphi = \int_{B_1} f\varphi - cu\varphi + (a_{ij}(x_0) - a_{ij}(x)) D_i u D_j \varphi.$$

In $B_r(x_0)$ the Dirichlet problem

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i w D_j \varphi = 0 \quad \text{for any} \quad \varphi \in H_0^1(B_r(x_0))$$

has a unique solution with $w$ with $w - u \in H_0^1(B_r(x_0))$. Obviously the function $v = u - w \in H_0^1(B_r(x_0))$ satisfies the equation

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i v D_j \varphi = \int_{B_r(x_0)} f\varphi - cu\varphi + (a_{ij}(x_0) - a_{ij}(x)) D_i u D_j \varphi$$
$$\text{for any } \varphi \in H_0^1(B_r(x_0)).$$

By taking the test function $\varphi = v$ we obtain

$$\int_{B_r(x_0)} |Dv|^2 \le c\Big\{ \tau^2(r) \int_{B_r(x_0)} |Du|^2 + \left( \int_{B_r(x_0)} |c|^n \right)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 + \left( \int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \Big\}$$

where we used the Sobolev's inequality

$$\left( \int_{B_r(x_0)} v^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \le c(n) \left( \int_{B_r(x_0)} |Dv|^2 \right)^{\frac{1}{2}}$$

for $v \in H_0^1(B_r(x_0))$. Therefore Corollary 2.3 implies for any $0 < \rho \le r$

$$\int_{B_\rho(x_0)} |Du|^2 \le c\Big\{ \Big[ \left(\frac{\rho}{r}\right)^n + \tau^2(r) \Big] \int_{B_r(x_0)} |Du|^2 + \left( \int_{B_r(x_0)} |c|^n \right)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 + \left( \int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \Big\} \tag{1}$$

where $c$ is a positive constant depending only on $\lambda$ and $\Lambda$. By Hölder inequality there holds

$$\left( \int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \le \left( \int_{B_r(x_0)} |f|^q \right)^{\frac{2}{q}} r^{n-2+2\alpha}$$

where $\alpha = 2 - n/q \in (0,1)$ if $n/2 < q < n$. Hence (1) implies for any $B_r(x_0) \subset B_1$ and any $0 < \rho \le r$

$$\int_{B_\rho(x_0)} |Du|^2 \le C\Big\{ \Big[\Big(\frac{\rho}{r}\Big)^n + \tau^2(r)\Big] \int_{B_r(x_o)} |Du|^2 + r^{n-2+2\alpha}\|f\|^2_{L^q(B_1)} + \Big(\int_{B_r(x_0)} |c|^n\Big)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 \Big\}.$$

**Case 1.** $c \equiv 0$.
We have for any $B_r(x_0) \subset B_1$ and for any $0 < \rho \le r$

$$\int_{B_\rho(x_0)} |Du|^2 \le C\left\{ \Big[\Big(\frac{\rho}{r}\Big)^n + \tau^2(r)\Big] \int_{B_r(x_0)} |Du|^2 + r^{n-2+2\alpha}\|f\|^2_{L^q(B_1)} \right\}.$$

Now the result would follow if in the above inequality we could write $\rho^{n-2+2\alpha}$ instead of $r^{n-2+2\alpha}$. This is in fact true and is stated in the Lemma 1.4. By Lemma 1.4, there exists an $R_0 > 0$ such that for any $x_0 \in B_{\frac{1}{2}}$ and any $0 < \rho < r \le R_0$ we have

$$\int_{B_\rho(x_0)} |Du|^2 \le C\left\{ \Big(\frac{\rho}{r}\Big)^{n-2+2\alpha} \int_{B_r(x_0)} |Du|^2 + \rho^{n-2+2\alpha}\|f\|^2_{L^q(B_1)} \right\}.$$

In particular, taking $r = R_0$ yields for any $\rho < R_0$

$$\int_{B_\rho(x_0)} |Du|^2 \le C\rho^{n-2+2\alpha} \left\{ \int_{B_1} |Du|^2 + \|f\|^2_{L^q(B_1)} \right\}.$$

**Case 2.** General case. We have for any $B_r(x_0) \subset B_1$ and any $0 < \rho \le r$

$$\int_{B_\rho(x_0)} |Du|^2 \le C\left\{ \Big[\Big(\frac{\rho}{r}\Big)^n + \tau^2(r)\Big] \int_{B_r(x_0)} |Du|^2 + r^{n-2+2\alpha}\chi(F) + \int_{B_r(x_0)} u^2 \right\} \tag{2}$$

where $\chi(F) = \|f\|^2_{L^q(B_1)}$. We will prove for any $x_0 \in B_{1/2}$ and any $0 < \rho < r \le 1/2$

$$\int_{B_\rho(x_0)} |Du|^2 \le C\Big\{ \Big[\Big(\frac{\rho}{r}\Big)^n + \tau^2(r)\Big] \int_{B_r(x_0)} |Du|^2 + r^{n-2+2\alpha}\Big[\chi(F) + \int_{B_1} u^2 + \int_{B_1} |Du|^2\Big] \Big\}. \tag{3}$$

We need a bootstrap argument. First by Lemma 1.3, there exists an $R_1 \in (1/2, 1)$ such that there holds for any $x_0 \in B_{R_1}$ and any $0 < r \leq 1 - R_1$

$$\int_{B_r(x_0)} u^2 \leq Cr^{\delta_1} \left\{ \int_{B_1} |Du|^2 + \int_{B_1} u^2 \right\} \tag{4}$$

where $\delta_1 = 2$ if $n > 2$ and $\delta_1$ is arbitrary in (0,2) if $n = 2$. This, with (2), yields

$$\int_{B_\rho(x_0)} |Du|^2 \leq C \left\{ \left[ \left( \frac{\rho}{r} \right)^n + \tau^2(r) \right] \int_{B_r(x_0)} |Du|^2 + r^{n-2+2\alpha} \chi(F) + r^{\delta_1} \|u\|^2_{H^1(B_1)} \right\}.$$

Then (3) holds in the following cases:

(i) $n = 2$, by choosing $\delta_1 = 2\alpha$;

(ii) $n > 2$ while $n - 2 + 2\alpha \leq 2$, by choosing $\delta_1 = 2$.

For $n > 2$ and $n - 2 + 2\alpha > 2$, we have

$$\int_{B_\rho(x_0)} |Du|^2 \leq c \left\{ \left[ \left( \frac{\rho}{r} \right)^n + \tau^2(r) \right] \int_{B_r(x_0)} |Du|^2 + r^2 [\chi(F) + \|u\|^2_{H^1(B_1)}] \right\}.$$

Lemma 1.4 again yields for any $x_0 \in B_{R_1}$ and any $0 < r \leq 1 - R_1$

$$\int_{B_r(x_0)} |Du|^2 \leq Cr^2 \left\{ \chi(F) + \|u\|^2_{H^1(B_1)} \right\}.$$

Hence by Lemma 1.3, there exists an $R_2 \in (1/2, R_1)$ such that there holds for any $x_0 \in B_{R_2}$ and any $0 < r \leq R_1 - R_2$

$$\int_{B_r(x_0)} u^2 \leq Cr^{\delta_2} \left\{ \chi(F) + \|u\|^2_{H^1(B_1)} \right\} \tag{5}$$

where $\delta_2 = 4$ if $n > 4$ and $\delta_2$ is arbitrary in $(2, n)$ if $n = 3$ or 4. Notice (5) is an improvement compared with (4). Substitute (5) in (2) and continue the process. After finite steps, we get (3).

This finishes the proof.

## §3. Hölder Continuity of Gradients

In this section we will prove Hölder regularity for gradients of Solutions. We follow the same idea used to prove Theorem 2.1.

Suppose $a_{ij} \in L^\infty(B_1)$ is uniformly elliptic in $B_1 = B_1(0)$, i.e.,

$$\lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \text{ for any } x \in B_1, \xi \in \mathbb{R}^n.$$

We assume that $u \in H^1(B_1)$ satisfies

$$\int_{B_1} a_{ij} D_i u D_j \varphi + cu\varphi = \int_{B_1} f\varphi \quad \text{for any } \varphi \in H^1_0(B_1). \tag{$*$}$$

The main theorems we will prove are the following Hölder estimates for gradients.

**Theorem 3.1.** *Let $u \in H^1(B_1)$ solve (*). Assume $a_{ij} \in C^\alpha(\bar{B}_1)$, $c \in L^q(B_1)$ and $f \in L^q(B_1)$ for some $q > n$ and $\alpha = 1 - n/q \in (0,1)$. Then $Du \in C^\alpha(B_1)$. Moreover, there exists an $R_0 = R_0(\lambda, |a_{ij}|_{C^\alpha}, |c|_{L^q})$ such that for any $x \in B_{\frac{1}{2}}$ and $r \le R_0$ there holds*

$$\int_{B_r(x)} |Du - (Du)_{x,r}|^2 \le C r^{n+2\alpha} \{ \|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)} \}$$

*where $C = C(\lambda, |a_{ij}|_{C^\alpha}, |c|_{L^q})$ is a positive constant.*

*Proof.* We shall decompose $u$ into a sum $v+w$ where $w$ satisfies a homogeneous equation and $v$ has estimates in terms of nonhomogeneous terms.

For any $B_r(x_0) \subset B_1$ write the equation in the following form

$$\int_{B_1} a_{ij}(x_0) D_i u D_j \varphi = \int_{B_1} f\varphi - cu\varphi + (a_{ij}(x_0) - a_{ij}(x)) D_i u D_j \varphi.$$

In $B_r(x_0)$ the Dirichlet problem

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i w D_j \varphi = 0 \quad \text{for any} \quad \varphi \in H^1_0(B_r(x_0))$$

has a unique solution $w$ with $w - u \in H^1_0(B_r(x_0))$. Obviously the function $v = u - w \in H^1_0(B_r(x_0))$ satisfies the equation

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i v D_j \varphi = \int_{B_r(x_0)} f\varphi - cu\varphi + (a_{ij}(x_0) - a_{ij}(x)) D_i u D_j \varphi \qquad \text{for any } \varphi \in H^1_0(B_r(x_0)).$$

By taking the test function $\varphi = v$ we obtain

$$\int_{B_r(x_0)} |Dv|^2 \le c\Big\{ \tau^2(r) \int_{B_r(x_0)} |Du|^2 + \Big( \int_{B_r(x_0)} |c|^n \Big)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 + \Big( \int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \Big)^{\frac{n+2}{n}} \Big\}.$$

Therefore Corollary 2.3 implies for any $0 < \rho \le r$

(1)
$$\int_{B_\rho(x_0)} |Du|^2 \le c\Big\{ \Big[ \Big(\frac{\rho}{r}\Big)^n + \tau^2(r) \Big] \int_{B_r(x_0)} |Du|^2 + \Big( \int_{B_r(x_0)} |c|^n \Big)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 + \Big( \int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \Big)^{\frac{n+2}{n}} \Big\},$$

and

$$\begin{aligned}
(2) \qquad & \int_{B_\rho(x_0)} |Du-(Du)_{x_0,\rho}|^2 \le c\Big\{ \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du-(Du)_{x_0,r}|^2 \\
& +\tau^2(r)\int_{B_r(x_0)} |Du|^2 + \Big(\int_{B_r(x_0)} |c|^n\Big)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 + \Big(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\Big)^{\frac{n+2}{n}} \Big\}
\end{aligned}$$

where $c$ is a positive constant depending only on $\lambda$ and $\Lambda$.

By Hölder inequality we have for any $B_r(x_0) \subset B_1$

$$\Big(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\Big)^{\frac{n+2}{n}} \le \Big(\int_{B_r(x_0)} |f|^q\Big)^{\frac{2}{q}} r^{n+2\alpha},$$

and

$$\Big(\int_{B_r(x_0)} |c|^n\Big)^{\frac{2}{n}} \le r^{2\alpha} \Big(\int_{B_r(x_0)} |c|^q\Big)^{\frac{2}{q}}$$

with $\alpha = 1 - n/q$.

**Case 1.** $a_{ij} \equiv$ const., $c \equiv 0$.

In this case $\tau(r) \equiv 0$. Hence by estimate (2) there holds for any $B_r(x_0) \subset B_1$ and $0 < \rho \le r$,

$$\int_{B_\rho(x_0)} |Du-(Du)_{x_0,\rho}|^2 \le C\Big\{ \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du-(Du)_{x_0,r}|^2 + r^{n+2\alpha}\|f\|^2_{L^q(B_1)}\Big\}.$$

By Lemma 1.4, we may replace $r^{n+2\alpha}$ by $\rho^{n+2\alpha}$ to get the result.

**Case 2.** $c \equiv 0$.

By (1) and (2), we have for any $B_r(x_0) \subset B_1$ and any $\rho < r$

$$(3) \qquad \int_{B_\rho(x_0)} |Du|^2 \le C\left\{ \Big[\Big(\frac{\rho}{r}\Big)^n + r^{2\alpha}\Big] \int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha}\|f\|^2_{L^q(B_1)}\right\}$$

and

$$\begin{aligned}
(4) \qquad & \int_{B_\rho(x_0)} |Du-(Du)_{x_0,\rho}|^2 \le C\Big\{ \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du-(Du)_{x_0,r}|^2 \\
& \qquad + r^{2\alpha}\int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha}\|f\|^2_{L^q(B_1)}\Big\}.
\end{aligned}$$

We need to estimate the integral

$$\int_{B_r(x_0)} |Du|^2.$$

Write $\chi(F) = \|f\|^2_{L^q(B_1)}$.

Take small $\delta > 0$. Then (3) implies

$$\int_{B_\rho(x_0)} |Du|^2 \le C\left\{\left[\left(\frac{\rho}{r}\right)^n + r^{2\alpha}\right]\int_{B_r(x_0)} |Du|^2 + r^{n-2\delta}\chi(F)\right\}.$$

Hence Lemma 1.4 implies the existence of an $R_1 \in (3/4, 1)$ with $r_1 = 1 - R_1$ such that for any $x_0 \in B_{R_1}$ and any $0 < r \le r_1$ there holds

$$\int_{B_r(x_0)} |Du|^2 \le Cr^{n-2\delta}\left\{\chi(F) + \|Du\|^2_{L^2(B_1)}\right\}. \tag{5}$$

Therefore by substituting (5) in (4) we obtain for any $0 < \rho < r \le r_1$

$$\begin{aligned}\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\Bigg\{ \left(\frac{\rho}{r}\right)^{n+2} &\int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 \\ &+ r^{n+2\alpha-2\delta}\left[\chi(F) + \|Du\|^2_{L^2(B_1)}\right]\Bigg\}.\end{aligned}$$

By Lemma 1.4 again, there holds for any $x_0 \in B_{R_1}$ and any $0 < \rho < r \le r_1$

$$\begin{aligned}\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\Bigg\{ \left(\frac{\rho}{r}\right)^{n+2\alpha-2\delta} &\int_{B_r} |Du - (Du)_{x_0,r}|^2 \\ &+ \rho^{n+2\alpha-2\delta}[\chi(F) + \|Du\|^2_{L^2(B_1)}]\Bigg\}.\end{aligned}$$

With $r = r_1$ this implies that for any $x_0 \in B_{R_1}$ and any $0 < r \le r_1$

$$\int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 \le Cr^{n+2\alpha-2\delta}\left\{\chi(F) + \|Du\|^2_{L^2(B_1)}\right\}.$$

Hence $Du \in C^{\alpha-\delta}_{\text{loc}}$ for any $\delta > 0$ small. In particular $Du \in L^\infty_{\text{loc}}$ and there holds

$$\boxed{\sup_{B_{\frac{3}{4}}} |Du|^2 \le C\left\{\chi(F) + \|Du\|^2_{L^2(B_1)}\right\}.} \tag{6}$$

Combining (4) and (6), there holds for any $x_0 \in B_{\frac{1}{2}}$ and $0 < \rho < r \le r_1$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\Big\{ \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + r^{n+2\alpha}\Big[\chi(F) + \|Du\|^2_{L^2(B_1)}\Big] \Big\}.$$

By Lemma 1.4 again, this implies

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\Big\{ \Big(\frac{\rho}{r}\Big)^{n+2\alpha} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + \rho^{n+2\alpha}\Big[\chi(F) + \|Du\|^2_{L^2(B_1)}\Big] \Big\}.$$

Choose $r = r_1$. We have for any $x_0 \in B_{\frac{1}{2}}$ and $r \le r_1$

$$\int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 \le c r^{n+2\alpha} \Big\{\chi(F) + \|Du\|^2_{L^2(B_1)}\Big\}.$$

**Case 3.** General case. By (1) and (2) we have for any $B_r(x_0) \subset B_1$ and $\rho < r$

$$\int_{B_\rho(x_0)} |Du|^2 \le C\left\{\left[\Big(\frac{\rho}{r}\Big)^n + r^{2\alpha}\right]\int_{B_r(x_0)} |Du|^2 + \int_{B_r(x_0)} u^2 + r^{n+2\alpha}\chi(F)\right\}, \tag{7}$$

and

$$\begin{aligned}\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\left\{\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 \right.\\ \left. + r^{2\alpha}\left[\int_{B_r(x_0)} u^2 + \int_{B_r(x_0)} |Du|^2\right] + r^{n+2\alpha}\chi(F)\right\},\end{aligned} \tag{8}$$

where $\chi(F) = \|f\|^2_{L^q(B_1)}$.

In (7), we may replace $r^{n+2\alpha}$ by $r^n$. As in the proof of Theorem 2.1, we may show that for any small $\delta > 0$ there exists an $R_1 \in (3/4, 1)$ such that for any $x \in B_{R_1}$ and $r < 1 - R_1$

$$\int_{B_r(x_0)} |Du|^2 \le C r^{n-2\delta} \Big\{\chi(F) + \|u\|^2_{H^1(B)}\Big\}. \tag{9}$$

By Lemma 1.3, we also get

$$\int_{B_r(x_0)} u^2 \le C r^{n-2\delta} \left\{ \chi(F) + \|u\|^2_{H^1(B)} \right\}. \tag{10}$$

Write

$$\chi(F,u) = \|f\|^2_{L^q} + \|u\|^2_{H^1}.$$

Then (8), (9), (10) imply that

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le c \left\{ \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + r^{n+2\alpha-2\delta}\chi(F,u) \right\}.$$

Hence Lemma 1.4 and Theorem 1.1 imply that $Du \in C^{\alpha-\delta}_{\text{loc}}$ for small $\delta < \alpha$. In particular $u \in C^1_{\text{loc}}$ with the estimate

$$\sup_{B_{\frac{3}{4}}} |u|^2 + \sup_{B_{\frac{3}{4}}} |Du|^2 \le C\chi(F,u). \tag{11}$$

Now (8) and (11) imply that

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C \left\{ \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + r^{n+2\alpha}\chi(F,u) \right\}.$$

This finishes the proof of Theorem 3.1.

*Remark.* It is natural to ask whether $f \in L^\infty(B_1)$, with appropriate assumptions on $a_{ij}$ and $c$, implies $Du \in C^1_{loc}$. Consider a special case

$$\int_{B_1} D_i u D_i \varphi = \int_{B_1} f\varphi \quad \text{for any} \quad \varphi \in H^1_0(B_1).$$

There exists an example showing that $f \in C$ and $u \in C^{1,\alpha}_{loc}$ for any $\alpha \in (0,1)$ while $D^2 u \notin C$.

*Example.* In the $n$-dimensional ball $B_R = B_R(0)$ of radius $R < 1$ consider

$$\triangle u = \frac{x_2^2 - x_1^2}{2|x|^2} \left\{ \frac{n+2}{(-\log|x|)^{1/2}} + \frac{1}{2(-\log|x|)^{3/2}} \right\}$$

where the right side is continuous in $\bar{B}_R$ if we set it equal to zero at the origin. The function $u(x) = (x_1^2 - x_2^2)(-\log|x|)^{1/2} \in C(\bar{B}_R) \cap C^\infty(\bar{B}_R \setminus \{0\})$ satisfies the above equation in $B_R \setminus \{0\}$ and the boundary condition $u = \sqrt{-\log R}(x_1^2 - x_2^2)$ on $\partial B_R$. But $u$ cannot be a classical solution of the problem since $\lim_{|x|\to 0} D_{11}u = \infty$ and therefore $u$ is not in $C^2(B_R)$. In fact the problem has no classical solution (although it has a weak solution). Assume on the contrary that a classical solution $v$ exists. Then the function $w = u - v$ is harmonic and bounded in $B_R \setminus \{0\}$. By a theorem from harmonic function theory on removable singularities, $w$ may be redefined at the origin so that $\triangle w = 0$ in $B_R$ and therefore belongs to $C^2(B_R)$. In particular, the (finite) limit $\lim_{|x|\to 0} D_{11}u$ exists, which is a contradiction.

## CHAPTER 4
## WEAK SOLUTIONS, PART II
## GUIDE

This chapter covers well-know theory of De Giorgi-Nash-Moser. We presented both approaches of De Giorgi and Moser for the purpose that students can make comparisons and can see the ideas involved are essentially the same.The classical paper [LSW] is certainly very a nice material for further readings.One may also wish to compare the results in [LSW] and [GS].

In this chapter we continue the discussion of the regularity theory for weak solutions to elliptic equations of divergence form. We will focus on the DeGiorgi-Nash-Moser theory.

### §1. Local Boundedness

In the following three sections we will discuss the DeGiorgi-Nash-Moser theory for linear elliptic equations. In this section we will prove the local boundedness of solutions. In the next section we will prove Hölder continuity. Then in Section 3 we will discuss the Harnack inequality. For all results in these three sections there is no regularity assumption of coefficients.

The main theorem of this section is the following boundedness result.

**Theorem 1.1.** *Suppose $a_{ij} \in L^\infty(B_1)$ and $c \in L^q(B_1)$ for some $q > n/2$ satisfy the following assumptions*

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \text{ for any } x \in B_1, \xi \in \mathbb{R}^n,$$

*and*

$$|a_{ij}|_{L^\infty} + \|c\|_{L^q} \leq \Lambda$$

*for some positive constants $\lambda$ and $\Lambda$. Suppose that $u \in H^1(B_1)$ is a subsolution in the following sense*

$$(*) \qquad \int_{B_1} a_{ij}D_iuD_j\varphi + cu\varphi \leq \int_{B_1} f\varphi \quad \text{for any} \quad \varphi \in H_0^1(B_1) \text{ and } \varphi \geq 0 \text{ in } B_1.$$

*If $f \in L^q(B_1)$, then $u^+ \in L^\infty_{loc}(B_1)$. Moreover there holds for any $\theta \in (0,1)$ and any $p > 0$*

$$\sup_{B_\theta} u^+ \leq C\left\{\frac{1}{(1-\theta)^{n/p}}\|u^+\|_{L^p(B_1)} + \|f\|_{L^q(B_1)}\right\}$$

*where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant.*

In the following we use two approaches to prove this theorem, one by DeGiorgi and the other by Moser.

*Proof.* We first prove for $\theta = 1/2$ and $p = 2$.

**Method 1. Approach by DeGiorgi.**

Consider $v = (u-k)^+$ for $k \geq 0$ and $\zeta \in C_0^1(B_1)$. Set $\varphi = v\zeta^2$ as the test function. Note $v = u - k, Dv = Du$ a.e. in $\{u > k\}$ and $v = 0, Dv = 0$ a.e. in $\{u \leq k\}$. Hence if we substitute such defined $\varphi$ in $(*)$, we integrate in the set $\{u > k\}$.

By Hölder inequality we have

$$\begin{aligned}\int a_{ij} D_i u D_j \varphi &= \int a_{ij} D_i u D_j v \zeta^2 + 2a_{ij} D_i u D_j \zeta v \zeta \\ &\geq \lambda \int |Dv|^2 \zeta^2 - 2\Lambda \int |Dv||D\zeta| v\zeta \\ &\geq \frac{\lambda}{2} \int |Dv|^2 \zeta^2 - \frac{2\Lambda^2}{\lambda} \int |D\zeta|^2 v^2.\end{aligned}$$

Hence we obtain

$$\int |Dv|^2 \zeta^2 \leq C \Big\{ \int v^2 |D\zeta|^2 + \int |c| v^2 \zeta^2 + k^2 \int |c| \zeta^2 + \int |f| v \zeta^2 \Big\}$$

from which the estimate

$$\int |D(v\zeta)|^2 \leq C \Big\{ \int v^2 |D\zeta|^2 + \int |c| v^2 \zeta^2 + k^2 \int |c| \zeta^2 + \int |f| v \zeta^2 \Big\}$$

follows.

Recall the Sobolev inequality for $v\zeta \in H_0^1(B_1)$

$$\Big( \int_{B_1} (v\zeta)^{2^*} \Big)^{\frac{2}{2^*}} \leq c(n) \int_{B_1} |D(v\zeta)|^2$$

where $2^* = 2n/(n-2)$ for $n > 2$ and $2^* > 2$ is arbitrary if $n = 2$. Hölder inequality implies that with $\delta > 0$ small and $\zeta \leq 1$

$$\begin{aligned}\int |f| v\zeta^2 &\leq \Big( \int |f|^q \Big)^{\frac{1}{q}} \Big( \int |v\zeta|^{2^*} \Big)^{\frac{1}{2^*}} |\{v\zeta \neq 0\}|^{1 - \frac{1}{2^*} - \frac{1}{q}} \\ &\leq c(n) \|f\|_{L^q} \Big( \int |D(v\zeta)|^2 \Big)^{\frac{1}{2}} |\{v\zeta \neq 0\}|^{\frac{1}{2} + \frac{1}{n} - \frac{1}{q}} \\ &\leq \delta \int |D(v\zeta)|^2 + c(n, \delta) \|f\|_{L^q}^2 |\{v\zeta \neq 0\}|^{1 + \frac{2}{n} - \frac{2}{q}}.\end{aligned}$$

Note $1 + \frac{2}{n} - \frac{2}{q} > 1 - \frac{1}{q}$ if $q > n/2$. Therefore we have the following estimate

$$\int |D(v\zeta)|^2 \le C\Big\{ \int v^2|D\zeta|^2 + \int |c|v^2\zeta^2 + k^2 \int |c|\zeta^2 + F^2|\{v\zeta \neq 0\}|^{1-\frac{1}{q}} \Big\}$$

where $F = \|f\|_{L^q(B_1)}$.

We claim that there holds

$$\int |D(v\zeta)|^2 \le C\Big\{ \int v^2|D\zeta|^2 + (k^2 + F^2)|\{v\zeta \neq 0\}|^{1-\frac{1}{q}} \Big\} \tag{1}$$

if $|\{v\zeta \neq 0\}|$ is small.

It is obvious if $c \equiv 0$. In fact in this special case there is no restriction on the set $\{v\zeta \neq 0\}$. In general, Hölder inequality implies that

$$\begin{aligned}\int |c|v^2\zeta^2 &\le \Big(\int |c|^q\Big)^{\frac{1}{q}} \Big(\int (v\zeta)^{2^*}\Big)^{\frac{2}{2^*}} |\{v\zeta \neq 0\}|^{1-\frac{2}{2^*}-\frac{1}{q}} \\ &\le c(n) \int |D(v\zeta)|^2 \Big(\int |c|^q\Big)^{\frac{1}{q}} |\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{1}{q}},\end{aligned}$$

and

$$\int |c|\zeta^2 \le \Big(\int |c|^q\Big)^{\frac{1}{q}} |\{v\zeta \neq 0\}|^{1-\frac{1}{q}}.$$

Therefore we have

$$\begin{aligned}\int |D(v\zeta)|^2 \le C\Big\{ &\int v^2|D\zeta|^2 + \int |D(v\zeta)|^2|\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{1}{q}} \\ &+ (k^2 + F^2)|\{v\zeta \neq 0\}|^{1-\frac{1}{q}} \Big\}.\end{aligned}$$

This implies (1) if $|\{v\zeta \neq 0\}|$ is small.

To continue we obtain by Sobolev inequality

$$\begin{aligned}\int (v\zeta)^2 &\le \Big(\int (v\zeta)^{2^*}\Big)^{\frac{2}{2^*}} |\{v\zeta \neq 0\}|^{1-\frac{2}{2^*}} \\ &\le c(n) \int |D(v\zeta)|^2|\{v\zeta \neq 0\}|^{\frac{2}{n}}.\end{aligned}$$

Therefore we have

$$\int (v\zeta)^2 \le C\Big\{ \int v^2|D\zeta|^2|\{v\zeta \neq 0\}|^{\frac{2}{n}} + (k + F)^2|\{v\zeta \neq 0\}|^{1+\frac{2}{n}-\frac{1}{q}} \Big\}$$

if $|\{v\zeta \neq 0\}|$ is small. Hence there exists an $\varepsilon > 0$ such that

$$\int (v\zeta)^2 \leq C\Big\{ \int v^2 |D\zeta|^2 |\{v\zeta \neq 0\}|^{\varepsilon} + (k+F)^2 |\{v\zeta \neq 0\}|^{1+\varepsilon} \Big\}$$

if $|\{v\zeta \neq 0\}|$ is small. Choose the cut-off function in the following way. For any fixed $0 < r < R \leq 1$ choose $\zeta \in C_0^\infty(B_R)$ such that $\zeta \equiv 1$ in $B_r$ and $0 \leq \zeta \leq 1$ and $|D\zeta| \leq 2(R-r)^{-1}$ in $B_1$. Set

$$A(k,r) = \{x \in B_r; u \geq k\}.$$

We conclude that for any $0 < r < R \leq 1$ and $k > 0$

$$(2) \quad \int_{A(k,r)} (u-k)^2 \leq C\Big\{ \frac{1}{(R-r)^2} |A(k,R)|^{\varepsilon} \int_{A(k,R)} (u-k)^2 + (k+F)^2 |A(k,R)|^{1+\varepsilon} \Big\}$$

if $|A(k,R)|$ is small. Note

$$|A(k,R)| \leq \frac{1}{k} \int_{A(k,R)} u^+ \leq \frac{1}{k} \|u^+\|_{L^2}.$$

Hence (2) holds if $k \geq k_0 = C\|u^+\|_{L^2}$ for some large $C$ depending only on $\lambda$ and $\Lambda$.

Next we would show that there exists some $k = C(k_0 + F)$ such that

$$\int_{A(k,1/2)} (u-k)^2 = 0.$$

To continue we take any $h > k \geq k_0$ and any $0 < r < 1$. It is obvious that $A(k,r) \supset A(h,r)$. Hence we have

$$\int_{A(h,r)} (u-h)^2 \leq \int_{A(k,r)} (u-k)^2$$

and

$$|A(h,r)| = |B_r \cap \{u - k > h - k\}| \leq \frac{1}{(h-k)^2} \int_{A(k,r)} (u-k)^2.$$

Therefore by (2) we have for any $h > k \geq k_0$ and $1/2 \leq r < R \leq 1$

$$\begin{aligned}\int_{A(h,r)} (u-h)^2 &\leq C\Big\{ \frac{1}{(R-r)^2} \int_{A(h,R)} (u-h)^2 + (h+F)^2 |A(h,R)| \Big\} |A(h,R)|^{\varepsilon} \\ &\leq C\Big\{ \frac{1}{(R-r)^2} + \frac{(h+F)^2}{(h-k)^2} \Big\} \frac{1}{(h-k)^{2\varepsilon}} \Big( \int_{A(k,R)} (u-k)^2 \Big)^{1+\varepsilon}\end{aligned}$$

or

$$\|(u-h)^+\|_{L^2(B_r)} \le C\left\{\frac{1}{R-r} + \frac{h+F}{h-k}\right\}\frac{1}{(h-k)^\varepsilon}\|(u-k)^+\|^{1+\varepsilon}_{L^2(B_R)}. \tag{3}$$

Now we carry out the iteration. Set $\varphi(k,r) = \|(u-k)^+\|_{L^2(B_r)}$. For $\tau = 1/2$ and some $k > 0$ to be determined, define for $\ell = 0, 1, 2, \cdots$,

$$k_\ell = k_0 + k(1 - \frac{1}{2^\ell}) \quad (\le k_0 + k)$$
$$r_\ell = \tau + \frac{1}{2^\ell}(1-\tau).$$

Obviously we have

$$k_\ell - k_{\ell-1} = \frac{k}{2^\ell}, \qquad r_{\ell-1} - r_\ell = \frac{1}{2^\ell}(1-\tau).$$

Therefore we have for $\ell = 0, 1, 2, \cdots$

$$\begin{aligned}\varphi(k_\ell, r_\ell) &\le C\left\{\frac{2^\ell}{1-\tau} + \frac{2^\ell(k_0+F+k)}{k}\right\}\frac{2^{\varepsilon\ell}}{k^\varepsilon}[\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon}\\ &\le \frac{C}{1-\tau}\cdot\frac{k_0+F+k}{k^{1+\varepsilon}}\cdot 2^{(1+\varepsilon)\ell}\cdot[\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon}.\end{aligned}$$

Next we prove inductively for any $\ell = 0, 1, \cdots$

$$\varphi(k_\ell, r_\ell) \le \frac{\varphi(k_0, r_0)}{\gamma^\ell} \quad \text{for some} \quad \gamma > 1 \tag{4}$$

if $k$ is sufficiently large. Obviously it is true for $\ell = 0$. Suppose true for $\ell - 1$. We write

$$[\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon} \le \left\{\frac{\varphi(k_0, r_0)}{\gamma^{\ell-1}}\right\}^{1+\varepsilon} = \frac{\varphi(k_0, r_0)^\varepsilon}{\gamma^{\ell\varepsilon-(1+\varepsilon)}}\cdot\frac{\varphi(k_0, r_0)}{\gamma^\ell}.$$

Then we obtain

$$\varphi(k_\ell, r_\ell) \le \boxed{\frac{C\gamma^{1+\varepsilon}}{1-\tau}\cdot\frac{k_0+F+k}{k^{1+\varepsilon}}\cdot[\varphi(k_0, r_0)]^\varepsilon\cdot\frac{2^{\ell(1+\varepsilon)}}{\gamma^{\ell\varepsilon}}}\cdot\frac{\varphi(k_0, r_0)}{\gamma^\ell}.$$

Choose $\gamma$ first such that $\gamma^\varepsilon = 2^{1+\varepsilon}$. Note $\gamma > 1$. Next, we need

$$\frac{C\gamma^{1+\varepsilon}}{1-\tau}\cdot\left(\frac{\varphi(k_0, r_0)}{k}\right)^\varepsilon\cdot\frac{k_0+F+k}{k} \le 1.$$

Therefore we choose

$$k = C_*\{k_0 + F + \varphi(k_0, r_0)\}$$

for $C_*$ large. Let $\ell \to +\infty$ in (4). We conclude

$$\varphi(k_0 + k, \tau) = 0.$$

Hence we have

$$\sup_{B_{\frac{1}{2}}} u^+ \le (C_* + 1)\{k_0 + F + \varphi(k_0, r_0)\}.$$

Recall $k_0 = C\|u^+\|_{L^2(B_1)}$ and $\varphi(k_0, r_0) \le \|u^+\|_{L^2(B_1)}$. This finishes the proof.

Next we give the second proof of Theorem 1.1.

**Method 2. Approach by Moser.** First we explain the idea. By choosing test function appropriately, we will estimate $L^{p_1}$ norm of $u$ in a smaller ball by $L^{p_2}$ norm of $u$ for $p_1 > p_2$ in a larger ball, i.e.,

$$\|u\|_{L^{p_1}(B_{r_1})} \le C\|u\|_{L^{p_2}(B_{r_2})}$$

for $p_1 > p_2$ and $r_1 < r_2$. This is a reversed Hölder inequality. As a sacrifice $C$ behaves like $\frac{1}{r_2 - r_1}$. By iteration and careful choice of $\{r_i\}$ and $\{p_i\}$, we will obtain the result.

For some $k > 0$ and $m > 0$, set $\bar{u} = u^+ + k$ and $\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m \\ k + m & \text{if } u \ge m \end{cases}$. Then we have $D\bar{u}_m = 0$ in $\{u < 0\}$ and $\{u > m\}$ and $\bar{u}_m \le \bar{u}$. Set the test function

$$\varphi = \eta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in H_0^1(B_1)$$

for some $\beta \ge 0$ and some nonnegative function $\eta \in C_0^1(B_1)$. Direct calculation yields

$$\begin{aligned} D\varphi &= \beta\eta^2\bar{u}_m^{\beta-1}D\bar{u}_m\bar{u} + \eta^2\bar{u}_m^\beta D\bar{u} + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}) \\ &= \eta^2\bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}) + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}). \end{aligned}$$

We should emphasize that later on we will begin the iteration with $\beta = 0$. Note $\varphi = 0$ and $D\varphi = 0$ in $\{u \le 0\}$. Hence if we substitute such $\varphi$ in the equation we integrate in the set $\{u > 0\}$. Note also that $u^+ \le \bar{u}$ and $\bar{u}_m^\beta\bar{u} - k^{\beta+1} \le \bar{u}_m^\beta\bar{u}$ for $k > 0$. First we have by Hölder inequality

$$\begin{aligned} \int a_{ij}D_iuD_j\varphi &= \int a_{ij}D_i\bar{u}(\beta D_j\bar{u}_m + D_j\bar{u})\eta^2\bar{u}_m^\beta + 2\int a_{ij}D_i\bar{u}D_j\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1})\eta \\ &\ge \lambda\beta\int\eta^2\bar{u}_m^\beta|D\bar{u}_m|^2 + \lambda\int\eta^2\bar{u}_m^\beta|D\bar{u}|^2 - \Lambda\int|D\bar{u}||D\eta|\bar{u}_m^\beta\bar{u}\eta \\ &\ge \lambda\beta\int\eta^2\bar{u}_m^\beta|D\bar{u}_m|^2 + \frac{\lambda}{2}\int\eta^2\bar{u}_m^\beta|D\bar{u}|^2 - \frac{2\Lambda^2}{\lambda}\int|D\eta|^2\bar{u}_m^\beta\bar{u}^2. \end{aligned}$$

Hence we obtain by noting $\bar{u} \geq k$

$$\begin{aligned}
&\beta \int \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \int \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 \\
\leq & C\Big\{ \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int \big(|c|\eta^2 \bar{u}_m^\beta \bar{u}^2 + |f|\eta^2 \bar{u}_m^\beta \bar{u} \Big\} \\
\leq & C\Big\{ \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int c_0 \eta^2 \bar{u}_m^\beta \bar{u}^2 \Big\},
\end{aligned}$$

where $c_0$ is defined as

$$c_0 = |c| + \frac{|f|}{k}.$$

Choose $k = \|f\|_{L^q}$ if $f$ is not identically zero. Otherwise choose arbitrary $k > 0$ and eventually let $k \to 0+$. By assumption we have

$$\|c_0\|_{L^q} \leq \Lambda + 1.$$

Set $w = \bar{u}_m^{\frac{\beta}{2}} \bar{u}$. Note

$$|Dw|^2 \leq (1+\beta)\{\beta \bar{u}_m^\beta |D\bar{u}_m|^2 + \bar{u}_m^\beta |D\bar{u}|^2\}.$$

Therefore we have

$$\int |Dw|^2 \eta^2 \leq C\Big\{(1+\beta)\int w^2 |D\eta|^2 + (1+\beta)\int c_0 w^2 \eta^2\Big\},$$

or

$$\int |D(w\eta)|^2 \leq C\Big\{(1+\beta)\int w^2 |D\eta|^2 + (1+\beta)\int c_0 w^2 \eta^2\Big\}.$$

Hölder inequality implies

$$\int c_0 w^2 \eta^2 \leq \Big(\int c_0^q\Big)^{\frac{1}{q}} \Big(\int (\eta w)^{\frac{2q}{q-1}}\Big)^{1-\frac{1}{q}} \leq (\Lambda+1)\Big(\int (\eta w)^{\frac{2q}{q-1}}\Big)^{1-\frac{1}{q}}.$$

By interpolation inequality and Sobolev inequality with $2^* = \frac{2n}{n-2} > \frac{2q}{q-1} > 2$ if $q > n/2$, we have

$$\begin{aligned}
\|\eta w\|_{L^{\frac{2q}{q-1}}} &\leq \varepsilon \|\eta w\|_{L^{2^*}} + C(n,q)\varepsilon^{-\frac{n}{2q-n}} \|\eta w\|_{L^2} \\
&\leq \varepsilon \|D(\eta w)\|_{L^2} + C(n,q)\varepsilon^{-\frac{n}{2q-n}} \|\eta w\|_{L^2}
\end{aligned}$$

for any small $\varepsilon > 0$. Therefore we obtain

$$\int |D(w\eta)|^2 \le C\left\{(1+\beta)\int w^2|D\eta|^2 + (1+\beta)^{\frac{2q}{2q-n}}\int w^2\eta^2\right\}$$

and in particular

$$\int |D(w\eta)|^2 \le C(1+\beta)^\alpha \int (|D\eta|^2+\eta^2)w^2,$$

where $\alpha$ in a positive number depending only on $n$ and $q$. Sobolev inequality then implies

$$\left(\int |\eta w|^{2\chi}\right)^{\frac{1}{\chi}} \le C(1+\beta)^\alpha \int (|D\eta|^2+\eta^2)w^2$$

where $\chi = \frac{n}{n-2} > 1$ for $n > 2$ and $\chi > 2$ for $n = 2$. Choose the cut-off function as follows. For any $0 < r < R \le 1$ set $\eta \in C_0^1(B_R)$ with the property

$$\eta \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\eta| \le \frac{2}{R-r}.$$

Then we obtain

$$\left(\int_{B_r} w^{2\chi}\right)^{\frac{1}{\chi}} \le C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} w^2.$$

Recalling the definition of $w$, we have

$$\left(\int_{B_r} \bar{u}^{2\chi}\bar{u}_m^{\beta\chi}\right)^{\frac{1}{\chi}} \le C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} \bar{u}^2\bar{u}_m^\beta.$$

Set $\gamma = \beta + 2 \ge 2$. Then we obtain

$$\left(\int_{B_r} \bar{u}_m^{\gamma\chi}\right)^{\frac{1}{\chi}} \le C\frac{(\gamma-1)^\alpha}{(R-r)^2}\int_{B_R} \bar{u}^\gamma$$

provided the integral in the right hand side is bounded. By letting $m \to +\infty$ we conclude that

$$\|\bar{u}\|_{L^{\gamma\chi}(B_r)} \le \left(C\frac{(\gamma-1)^\alpha}{(R-r)^2}\right)^{\frac{1}{\gamma}} \|\bar{u}\|_{L^\gamma(B_R)}$$

provided $\|\bar{u}\|_{L^\gamma(B_R)} < +\infty$, where $C = C(n,q,\lambda,\Lambda)$ is a positive constant independent of $\gamma$. The above estimate suggests that we iterate, beginning with $\gamma = 2$, as 2, $2\chi$, $2\chi^2$, $\cdots$. Now set for $i = 0, 1, 2, \cdots$,

$$\gamma_i = 2\chi^i \quad \text{and} \quad r_i = \frac{1}{2} + \frac{1}{2^{i+1}}.$$

By $\gamma_i = \chi\gamma_{i-1}$ and $r_{i-1} - r_i = 1/2^{i+1}$, we have for $i = 1, 2, \cdots$,

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le C(n, q, \lambda, \Lambda)^{\frac{i}{\chi^i}} \|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})}$$

provided $\|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})} < +\infty$. Hence by iteration we obtain

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \le C^{\sum \frac{i}{\chi^i}} \|\bar{u}\|_{L^2(B_1)}$$

in particular

$$\left( \int_{B_{\frac{1}{2}}} \bar{u}^{2\chi^i} \right)^{\frac{1}{2\chi^i}} \le C \left( \int_{B_1} \bar{u}^2 \right)^{\frac{1}{2}}.$$

Letting $i \to +\infty$ we get

$$\sup_{B_{\frac{1}{2}}} \bar{u} \le C\|\bar{u}\|_{L^2(B_1)}$$

or

$$\sup_{B_{\frac{1}{2}}} u^+ \le C\{\|u^+\|_{L^2(B_1)} + k\}.$$

Recall the definition of $k$. This finishes the proof for $p = 2$.

**Remark.** If the subsolution $u$ is bounded, we may simply take the test function

$$\varphi = \eta^2(\bar{u}^{\beta+1} - k^{\beta+1}) \in H_0^1(B_1)$$

for some $\beta \ge 0$ and some nonnegative function $\eta \in C_0^1(B_1)$.

Next we discuss the general case of Theorem 1.1. This is based on a dilation argument. Take any $R \le 1$. Define

$$\tilde{u}(y) = u(Ry) \text{ for } y \in B_1.$$

It is easy to see that $\tilde{u}$ satisfies the following equation

$$\int_{B_1} \tilde{a}_{ij} D_i\tilde{u} D_j\varphi + \tilde{c}\tilde{u}\varphi \le \int_{B_1} \tilde{f}\varphi$$
$$\text{for any } \quad \varphi \in H_0^1(B_1) \text{ and } \phi \ge 0 \text{ in } B_1$$

where

$$\tilde{a}(y) = a(Ry), \quad \tilde{c}(y) = R^2 c(Ry)$$

and

$$\tilde{f}(y) = R^2 f(Ry)$$

for any $y \in B_1$. Direct calculation shows

$$|\tilde{a}_{ij}|_{L^\infty(B_1)} + \|\tilde{c}\|_{L^q(B_1)} = |a_{ij}|_{L^\infty(B_R)} + R^{2-\frac{n}{q}}\|c\|_{L^q(B_R)} \le \Lambda.$$

We may apply what we just proved to $\tilde{u}$ in $B_1$ and rewrite the result in terms of $u$. Hence we obtain for $p \ge 2$

$$\sup_{B_{\frac{R}{2}}} u^+ \le C\{\frac{1}{R^{n/p}}\|u^+\|_{L^p(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\}$$

where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant.

The estimate in $B_{\theta R}$ can be obtained by applying the above result to $B_{(1-\theta)R}(y)$ for any $y \in B_{\theta R}$. Take $R = 1$. This is the Theorem 1.1 for any $\theta \in (0,1)$ and $p \ge 2$.

Now we prove the statement for $p \in (0,2)$. We showed that for any $\theta \in (0,1)$ and $0 < R \le 1$ there holds

$$\begin{aligned}\|u^+\|_{L^\infty(B_{\theta R})} &\le C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{2}}}\|u^+\|_{L^2(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\right\}\\ &\le C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{2}}}\|u^+\|_{L^2(B_R)} + \|f\|_{L^q(B_1)}\right\}.\end{aligned}$$

For $p \in (0,2)$ we have

$$\int_{B_R}(u^+)^2 \le \|u^+\|^{2-p}_{L^\infty(B_R)}\int_{B_R}(u^+)^p$$

and hence by Hölder inequality

$$\begin{aligned}&\|u^+\|_{L^\infty(B_{\theta R})}\\ \le& C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{2}}}\|u^+\|^{1-\frac{p}{2}}_{L^\infty(B_R)}\left(\int_{B_R}(u^+)^p dx\right)^{\frac{1}{2}} + \|f\|_{L^q(B_R)}\right\}\\ \le& \frac{1}{2}\|u^+\|_{L^\infty(B_R)} + C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{p}}}\left(\int_{B_R}(u^+)^p\right)^{\frac{1}{p}} + \|f\|_{L^q(B_R)}\right\}.\end{aligned}$$

Set $f(t) = \|u^+\|_{L^\infty(B_t)}$ for $t \in (0,1]$. Then for any $0 < r < R \le 1$

$$f(r) \le \frac{1}{2}f(R) + \frac{C}{(R-r)^{\frac{n}{p}}}\|u^+\|_{L^p(B_1)} + C\|f\|_{L^q(B_1)}.$$

We apply the following lemma to get for any $0 < r < R < 1$

$$f(r) \le \frac{C}{(R-r)^{\frac{n}{p}}} \|u^+\|_{L^p(B_1)} + C\|f\|_{L^q(B_1)}.$$

Let $R \to 1-$. We obtain for any $\theta < 1$

$$\|u^+\|_{L^\infty(B_\theta)} \le \frac{C}{(1-\theta)^{\frac{n}{p}}} \|u^+\|_{L^p(B_1)} + C\|f\|_{L^q(B_1)}.$$

We need the following simple lemma.

**Lemma 1.2.** *Let $f(t) \ge 0$ be bounded in $[\tau_0, \tau_1]$ with $\tau_0 \ge 0$. Suppose for $\tau_0 \le t < s \le \tau_1$ we have*

$$f(t) \le \theta f(s) + \frac{A}{(s-t)^\alpha} + B$$

*for some $\theta \in [0,1)$. Then for any $\tau_0 \le t < s \le \tau_1$ there holds*

$$f(t) \le c(\alpha, \theta) \left\{ \frac{A}{(s-t)^\alpha} + B \right\}.$$

*Proof.* Fix $\tau_0 \le t < s \le \tau_1$. For some $0 < \tau < 1$ we consider the sequence $\{t_i\}$ defined by

$$t_0 = t \text{ and } t_{i+1} = t_i + (1-\tau)\tau^i(s-t).$$

Note $t_\infty = s$. By iteration

$$f(t) = f(t_0) \le \theta^k f(t_k) + \left[ \frac{A}{(1-\tau)^\alpha}(s-t)^{-\alpha} + B \right] \sum_{i=0}^{k-1} \theta^i \tau^{-i\alpha}.$$

Choose $\tau < 1$ such that $\theta\tau^{-\alpha} < 1$, i.e., $\theta < \tau^\alpha < 1$. As $k \to \infty$ we have

$$f(t) \le c(\alpha, \theta) \left\{ \frac{A}{(1-\tau)^\alpha}(s-t)^{-\alpha} + B \right\}.$$

In the rest of this section we use Moser's iteration to prove a high integrability result, which is closely related to Theorem 1.1. For the next result we require $n \ge 3$.

**Theorem 1.3.** *Suppose* $a_{ij} \in L^\infty(B_1)$ *and* $c \in L^{n/2}(B_1)$ *satisfy the following assumption*

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \text{ for any } x \in B_1, \xi \in \mathbb{R}^n$$

*for some positive constants* $\lambda$ *and* $\Lambda$. *Suppose that* $u \in H^1(B_1)$ *is a subsolution in the following sense*

$$\int_{B_1} a_{ij}D_iuD_j\varphi + cu\varphi \leq \int_{B_1} f\varphi$$
$$\text{for any} \quad \varphi \in H_0^1(B_1) \text{ and } \varphi \geq 0 \text{ in } B_1.$$

*If* $f \in L^q(B_1)$ *for some* $q \in [\frac{2n}{n+2}, \frac{n}{2})$, *then* $u^+ \in L^{q^*}_{loc}(B_1)$ *for* $\frac{1}{q^*} = \frac{1}{q} - \frac{2}{n}$. *Moreover there holds*

$$\|u^+\|_{L^{q^*}(B_{\frac{1}{2}})} \leq C\Big\{\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\Big\}$$

*where* $C = C(n, \lambda, \Lambda, q, \varepsilon(K))$ *is a positive constant with*

$$\varepsilon(K) = \Big(\int_{\{|c|>K\}} |c|^{\frac{n}{2}}\Big)^{\frac{2}{n}}.$$

*Proof.* For $m > 0$, set $\bar{u} = u^+$ and $\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m \\ m & \text{if } u \geq m \end{cases}$. Then set the test function

$$\varphi = \eta^2 \bar{u}_m^\beta \bar{u} \in H_0^1(B_1)$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_0^1(B_1)$. By similar calculations as in the proof of Theorem 1.1 we conclude

$$\Big(\int \eta^{2\chi}\bar{u}_m^{\beta\chi}\bar{u}^{2\chi}\Big)^{\frac{1}{\chi}} \leq C(1+\beta)\Big\{\int |D\eta|^2\bar{u}_m^\beta\bar{u}^2 + \int |c|\eta^2\bar{u}_m^\beta\bar{u}^2 + \int |f|\eta^2\bar{u}_m^\beta\bar{u}\Big\}$$

where $\chi = \frac{n}{n-2} > 1$. Hölder inequality implies for any $K > 0$

$$\begin{aligned}\int |c|\eta^2\bar{u}_m^\beta\bar{u}^2 &\leq K\int_{\{|c|\leq K\}} \eta^2\bar{u}_m^\beta\bar{u}^2 + \int_{\{|c|>K\}} |c|\eta^2\bar{u}_m^\beta\bar{u}^2 \\ &\leq K\int \eta^2\bar{u}_m^\beta\bar{u}^2 + \Big(\int_{\{|c|>K\}} |c|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big(\int (\eta^2\bar{u}_m^\beta\bar{u}^2)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} \\ &\leq K\int \eta^2\bar{u}_m^\beta\bar{u}^2 + \varepsilon(K)\Big(\int \eta^{2\chi}\bar{u}_m^{\beta\chi}\bar{u}^{2\chi}\Big)^{\frac{1}{\chi}}.\end{aligned}$$

Note $\varepsilon(K) \to 0$ as $K \to +\infty$ since $c \in L^{n/2}(B_1)$. Hence for bounded $\beta$ we obtain by choosing large $K = K(\beta)$

$$\left(\int \eta^{2\chi}\bar{u}_m^{\beta\chi}\bar{u}^{2\chi}\right)^{\frac{1}{\chi}} \le C(1+\beta)\left\{\int(|D\eta|^2+\eta^2)\bar{u}_m^{\beta}\bar{u}^2 + \int |f|\eta^2\bar{u}_m^{\beta}\bar{u}\right\}.$$

Observe

$$\bar{u}_m^{\beta}\bar{u} \le \bar{u}_m^{\beta-\frac{\beta}{\beta+2}}\bar{u}^{1+\frac{\beta}{\beta+2}} = (\bar{u}_m^{\beta}\bar{u}^2)^{\frac{\beta+1}{\beta+2}}.$$

Therefore by Hölder inequality again we have for $\eta \le 1$

$$\begin{aligned}\int |f|\eta^2\bar{u}_m^{\beta}\bar{u} &\le \left(\int |f|^q\right)^{\frac{1}{q}}\left(\int (\eta^2\bar{u}_m^{\beta}\bar{u}^2)^{\chi}\right)^{\frac{\beta+1}{(\beta+2)\chi}}|\text{supp }\eta|^{1-\frac{1}{q}-\frac{\beta+1}{(\beta+2)\chi}}\\ &\le \varepsilon\left(\int \eta^{2\chi}\bar{u}^{\chi}\bar{u}_m^{\beta\chi}\right)^{\frac{1}{\chi}} + C(\varepsilon,\beta)\left(\int |f|^q\right)^{\frac{\beta+2}{q}},\end{aligned}$$

provided

$$1 - \frac{1}{q} - \frac{\beta+1}{(\beta+2)\chi} \ge 0$$

which is equivalent to

$$\beta + 2 \le \frac{q(n-2)}{n-2q}.$$

Hence $\beta$ is required to be bounded, depending only on $n$ and $q$. Then we obtain

$$\left(\int \eta^{2\chi}\bar{u}_m^{\beta\chi}\bar{u}^{2\chi}\right)^{\frac{1}{\chi}} \le C\left\{\int(|D\eta|^2+\eta^2)\bar{u}_m^{\beta}\bar{u}^2 + \|f\|_{L^q}^{\beta+2}\right\}.$$

By setting $\gamma = \beta + 2$, we have by definition of $q^*$

$$2 \le \gamma \le \frac{q(n-2)}{n-2q} = \frac{q^*}{\chi}. \tag{1}$$

We conclude, as before, for any such $\gamma$ in (1) and any $0 < r < R \le 1$

$$\|\bar{u}\|_{L^{\chi\gamma}(B_r)} \le C\left\{\frac{1}{(R-r)^{\frac{2}{\gamma}}}\|\bar{u}\|_{L^{\gamma}(B_R)} + \|f\|_{L^q(B_1)}\right\} \tag{2}$$

provided $\|\bar{u}\|_{L^{\gamma}(B_R)} < +\infty$. Again this suggests the iteration $2, 2\chi, 2\chi^2, \cdots$.

For given $q \in [\frac{2n}{n+2}, \frac{n}{2})$, there exists a positive integer $k$ such that

$$2\chi^{k-1} \leq \frac{q(n-2)}{n-2q} < 2\chi^k.$$

Hence for such $k$ we get by finitely many iterations of (2)

$$\|\bar{u}\|_{L^{2\chi^k}(B_{\frac{3}{4}})} \leq C\Big\{\|\bar{u}\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\Big\}$$

in particular

$$\|\bar{u}\|_{L^{\frac{q^*}{\chi}}(B_{\frac{3}{4}})} \leq C\Big\{\|\bar{u}\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\Big\}.$$

While with $\gamma = \frac{q^*}{\chi}$ in (2) we obtain

$$\|\bar{u}\|_{L^{q^*}(B_{\frac{1}{2}})} \leq C\Big\{\|\bar{u}\|_{L^{\frac{q^*}{\chi}}(B_{\frac{3}{4}})} + \|f\|_{L^q(B_1)}\Big\}.$$

This finishes the proof.

## §2. Hölder Continuity

We first discuss homogeneous equations with no lower order terms. Consider

$$Lu \equiv -D_i(a_{ij}(x)D_j u) \text{ in } B_1(0) \subset \mathbb{R}^n$$

where $a_{ij} \in L^\infty(B_1)$ satisfies

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } x \in B_1(0) \text{ and } \xi \in \mathbb{R}^n$$

for some positive constants $\lambda$ and $\Lambda$.

**Definition.** *The function $u \in H^1_{loc}(B_1)$ is called a subsolution (supersolution) of the equation*

$$Lu = 0$$

*if*

$$\int_{B_1} a_{ij}D_iuD_j\varphi \leq 0 (\geq 0)$$

*for all $\varphi \in H^1_0(B_1)$ and $\varphi \geq 0$.*

**Lemma 2.1.** *Let $\Phi \in C^{0,1}_{loc}(\mathbb{R})$ be convex. Then*

*(i) if $u$ is a subsolution and $\Phi' \geq 0$, then $v = \Phi(u)$ is also a subsolution provided $v \in H^1_{loc}(B_1)$.*

*(ii) if $u$ is a supersolution and $\Phi' \leq 0$, then $v = \Phi(u)$ is a subsolution provided $v \in H^1_{loc}(B_1)$.*

*Remark.* If $u$ is a subsolution, then $(u-k)^+$ is also a subsolution, where $(u-k)^+ = max\{0, u-k\}$. In this case $\Phi(s) = (s-k)^+$.

*Proof.* Direct computation.

(i) Assume first $\Phi \in C^2_{loc}(\mathbb{R})$. Then

$$\Phi'(s) \geq 0, \Phi''(s) \geq 0.$$

Consider $\varphi \in C^1_0(B_1)$ with $\varphi \geq 0$. Direct calculation yields

$$\begin{aligned}\int_{B_1} a_{ij} D_i v D_j \varphi &= \int_{B_1} a_{ij} \Phi'(u) D_i u D_j \varphi \\ &= \int_{B_1} a_{ij} D_i u D_j (\Phi'(u)\varphi) - \int_{B_1} (a_{ij} D_i u D_j u) \varphi \Phi''(u) \leq 0,\end{aligned}$$

where $\Phi'(u)\varphi \in H^1_0(B_1)$ is nonnegative. In general, set $\Phi_\epsilon(s) = \rho_\epsilon * \Phi(s)$ with $\rho_\epsilon$ as the standard mollifier. Then $\Phi'_\epsilon(s) = \rho_\epsilon * \Phi'(s) \geq 0$ and $\Phi''_\epsilon(s) \geq 0$. Hence $\Phi_\epsilon(u)$ is a subsolution by what we just proved. Note $\Phi'_\epsilon(s) \to \Phi'(s)$ a.e. as $\epsilon \to 0^+$. Hence Lebesgue dominant covergence theorem implies the result.

(ii) Similar.

We need the following Poincaré-Sobolev inequality.

**Lemma 2.2.** *For any $\epsilon > 0$ there exists a $C = C(\epsilon, n)$ such that for $u \in H^1(B_1)$ with*

$$|\{x \in B_1; u = 0\}| \geq \epsilon |B_1|$$

*there holds*

$$\int_{B_1} u^2 \leq C \int_{B_1} |Du|^2.$$

*Proof.* Suppose not. Then there exists a sequence $\{u_m\} \subset H^1(B_1)$ such that

$$|\{x \in B_1; u_m = 0\}| \geq \epsilon |B_1|$$

$$\int_{B_1} u_m^2 = 1 \quad \text{and} \quad \int_{B_1} |Du_m|^2 \to 0 \text{ as } m \to \infty.$$

Hence we may assume $u_m \to u_0 \in H^1(B_1)$ strongly in $L^2(B_1)$ and weakly in $H^1(B_1)$. Clearly $u_0$ is a nonzero constant. So

$$\begin{aligned} 0 = \lim_{m\to\infty} \int_{B_1} |u_m - u_0|^2 &\geq \lim_{m\to\infty} \int_{\{u_m=0\}} |u_m - u_0|^2 \\ &\geq |u_0|^2 \inf_m |\{u_m = 0\}| > 0. \end{aligned}$$

Contradiction.

**Theorem 2.3 (Density Theorem).** *Suppose $u$ is a positive supersolution in $B_2$ with*

$$|\{x \in B_1; u \geq 1\}| \geq \epsilon |B_1|.$$

*Then there exists a constant $C$ depending only on $\epsilon$, $n$ and $\Lambda/\lambda$ such that*

$$\inf_{B_{\frac{1}{2}}} u \geq C.$$

*Proof.* We may assume that $u \geq \delta > 0$. Then let $\delta \to 0+$.

By Lemma 2.1, $v = (\log u)^-$ is a subsolution, bounded by $\log \delta^{-1}$. Then Theorem 1.1 yields

$$\sup_{B_{\frac{1}{2}}} v \leq C \left( \int_{B_1} |v|^2 \right)^{\frac{1}{2}}.$$

Note $|\{x \in B_1; v = 0\}| = |\{x \in B_1; u \geq 1\}| \geq \epsilon |B_1|$. Lemma 2.2 implies

$$\sup_{B_{\frac{1}{2}}} v \leq C \left( \int_{B_1} |Dv|^2 \right)^{\frac{1}{2}}. \tag{1}$$

We will prove that the right-hand side is bounded. To this end, set test function as $\varphi = \frac{\zeta^2}{u}$ for $\zeta \in C_0^1(B_2)$. Then we obtain

$$0 \leq \int a_{ij} D_i u D_j \left( \frac{\zeta^2}{u} \right) = - \int \zeta^2 \frac{a_{ij} D_i u D_j u}{u^2} + 2 \int \frac{\zeta a_{ij} D_i u D_j \zeta}{u}$$

which implies

$$\int \zeta^2 |D \log u|^2 \leq C \int |D\zeta|^2.$$

So for fixed $\zeta \in C_0^1(B_2)$ with $\zeta \equiv 1$ in $B_1$ we have

$$\int_{B_1} |D \log u|^2 \leq C.$$

Combining with (1) we obtain

$$\sup_{B_{\frac{1}{2}}} v = \sup_{B_{\frac{1}{2}}} (\log u)^- \leq C$$

which gives

$$\inf_{B_{\frac{1}{2}}} u \geq e^{-C} > 0.$$

**Theorem 2.4 (Oscillation Theorem).** *Suppose that $u$ is a bounded solution of $Lu = 0$ in $B_2$. Then there exists a $\gamma = \gamma(n, \frac{\Lambda}{\lambda}) \in (0, 1)$ such that*

$$\underset{B_{\frac{1}{2}}}{osc}\, u \leq \gamma \underset{B_1}{osc}\, u.$$

*Proof.* In fact, local boundedness is proved in the previous section. Set

$$\alpha_1 = \sup_{B_1} u \quad \text{and} \quad \beta_1 = \inf_{B_1} u.$$

Consider the solution

$$\frac{u - \beta_1}{\alpha_1 - \beta_1} \quad \text{or} \quad \frac{\alpha_1 - u}{\alpha_1 - \beta_1}.$$

Note the following equivalence

$$u \geq \frac{1}{2}(\alpha_1 + \beta_1) \Longleftrightarrow \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{2}$$
$$u \leq \frac{1}{2}(\alpha_1 + \beta_1) \Longleftrightarrow \frac{\alpha_1 - u}{\alpha_1 - \beta_1} \geq \frac{1}{2}.$$

Case 1. Suppose that

$$|\{x \in B_1; \frac{2(u - \beta_1)}{\alpha_1 - \beta_1} \geq 1\}| \geq \frac{1}{2}|B_1|.$$

Apply the above theorem to $\frac{u-\beta_1}{\alpha_1-\beta_1} \geq 0$ in $B_1$. We have for some $C > 1$

$$\inf_{B_{\frac{1}{2}}} \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{C},$$

which results in the following estimate

$$\inf_{B_{\frac{1}{2}}} u \geq \beta_1 + \frac{1}{C}(\alpha_1 - \beta_1).$$

Case 2. Suppose

$$|\{x \in B_1; \frac{2(\alpha_1 - u)}{\alpha_1 - \beta_1} \geq 1\}| \geq \frac{1}{2}|B_1|.$$

Similarly we obtain

$$\sup_{B_{\frac{1}{2}}} u \leq \alpha_1 - \frac{1}{C}(\alpha_1 - \beta_1).$$

Now set

$$\alpha_2 = \sup_{B_{\frac{1}{2}}} u \quad \text{and} \quad \beta_2 = \inf_{B_{\frac{1}{2}}} u.$$

Note $\beta_2 \geq \beta_1, \alpha_2 \leq \alpha_1$. In both cases, we have

$$\alpha_2 - \beta_2 \leq (1 - \frac{1}{C})(\alpha_1 - \beta_1).$$

DeGiorgi theorem is an easy consequence of the above results.

**Theorem 2.5 (DeGiorgi).** *Suppose $Lu = 0$ weakly in $B_1$. Then there holds*

$$\sup_{B_{\frac{1}{2}}}|u(x)| + \sup_{x,y \in B_{\frac{1}{2}}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq c(n, \frac{\Lambda}{\lambda})\|u\|_{L^2(B_1)}$$

*with $\alpha = \alpha(n, \frac{\Lambda}{\lambda}) \in (0, 1)$.*

In the rest of the section we will discuss the Hölder continuity of solutions to general linear equations. We need the following lemma.

**Lemma 2.6.** *Suppose that $a_{ij} \in L^{\infty}(B_r)$ satisfies*

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \textit{for any } x \in B_r,\ \xi \in \mathbb{R}^n$$

*for some $0 < \lambda \leq \Lambda < +\infty$. Suppose $u \in H^1(B_r)$ satisfies*

$$\int_{B_r} a_{ij} D_i u D_j \varphi = 0 \qquad \textit{for any } \varphi \in H_0^1(B_r).$$

*Then there exists an* $\alpha \in (0,1)$ *such that for any* $\rho < r$ *there holds*

$$\int_{B_\rho} |Du|^2 \le C\left(\frac{\rho}{r}\right)^{n-2+2\alpha} \int_{B_r} |Du|^2$$

*where* $C$ *and* $\alpha$ *depend only on* $n$ *and* $\Lambda/\lambda$.

*Proof.* By dilation, consider $r = 1$. We restrict our consideration to the range $\rho \in \left(0, \frac{1}{4}\right]$, since it is trivial for $\rho \in (\frac{1}{4}, 1]$. We may further assume that $\int_{B_1} u = 0$ since the function $u - |B_1|^{-1}\int_{B_1} u$ solves the same equation. Poincaré inequality yields

$$\int_{B_1} u^2 \le c(n) \int_{B_1} |Du|^2.$$

Hence Theorem 2.5 implies for $|x| \le 1/2$

$$|u(x) - u(0)|^2 \le C|x|^{2\alpha} \int_{B_1} |Du|^2$$

where $\alpha \in (0,1)$ is determined in Theorem 2.5. For any $0 < \rho \le 1/4$ take a cut-off function $\zeta \in C_0^\infty(B_{2\rho})$ with $\zeta \equiv 1$ in $B_\rho$ and $0 \le \rho \le 1$ and $|D\zeta| \le 2/\rho$. Then set $\varphi = \zeta^2(u - u(0))$. Hence the equation yields

$$\begin{aligned} 0 &= \int_{B_1} a_{ij} D_i u \left(\zeta^2 D_j u + 2\zeta D_j \zeta (u - u(0))\right) \\ &\ge \frac{\lambda}{2} \int_{B_{2\rho}} \zeta^2 |Du|^2 - C \sup_{B_{2\rho}} |u - u(0)|^2 \int_{B_{2\rho}} |D\zeta|^2. \end{aligned}$$

Therefore we have

$$\int_{B_\rho} |Du|^2 \le C\rho^{n-2} \sup_{B_{2\rho}} |u - u(0)|^2.$$

The conclusion follows easily.

Now we may prove the following result in the same way we proved Theorem 2.1 in Chapter 3, with Lemma 2.2 in Chpater 3 replaced by Lemma 2.6.

**Theorem 2.7.** *Assume* $a_{ij} \in L^\infty(B_1)$ *and* $c \in L^n(B_1)$ *satisfies*

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \text{ for any } x \in B_1, \xi \in \mathbb{R}^n$$

*for some* $0 < \lambda \le \Lambda < +\infty$. *Suppose that* $u \in H^1(B_1)$ *satisfies*

$$\int_{B_1} a_{ij} D_j u D_i \varphi + cu\varphi = \int_{B_1} f\varphi \quad \text{for any } \varphi \in H_0^1(B_1).$$

*If $f \in L^q(B_1)$ for some $q > n/2$, then $u \in C^\alpha(B_1)$ for some $\alpha = \alpha(n, q, \lambda, \Lambda, \|c\|_{L^n}) \in (0,1)$. Moreover, there exists $R_0 = R_0(q, \lambda, \Lambda, \|c\|_{L^n})$ such that for any $x \in B_{\frac{1}{2}}$ and $r \le R_0$ there holds*

$$\int_{B_r(x)} |Du|^2 \le Cr^{n-2+2\alpha}\left\{\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)}\right\}$$

*where $C = C(n, q, \lambda, \Lambda, \|c\|_{L^n})$ is a positive constant.*

## §3. Moser's Harnack Inequality

In this section we only discuss equations without lower order terms. Suppose $\Omega$ is a domain in $\mathbb{R}^n$. We always assume that $a_{ij} \in L^\infty(\Omega)$ satisfies

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n$$

for some positive constants $\lambda$ and $\Lambda$.

**Theorem 3.1 (Local Boundedness).** *Let $u \in H^1(\Omega)$ be a nonnegative subsolution in $\Omega$ in the following sense*

$$\int_\Omega a_{ij}D_iuD_j\varphi \le \int_\Omega f\varphi \quad \textit{for any } \varphi \in H^1_0(\Omega) \textit{ and } \phi \ge 0 \textit{ in } \Omega.$$

*Suppose $f \in L^q(\Omega)$ for some $q > n/2$. Then there holds for any $B_R \subset \Omega$, any $0 < r < R$ and any $p > 0$*

$$\sup_{B_r} u \le C\{\frac{1}{(R-r)^{n/p}}\|u^+\|_{L^p(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\}$$

*where $C = C(n, \lambda, \Lambda, p, q)$ is a positive constant.*

*Proof.* This is a special case of Theorem 1.1 in the dilated version.

**Theorem 3.2 (Weak Harnack Inequality).** *Let $u \in H^1(\Omega)$ be a nonnegative supersolution in $\Omega$ in the following sense*

$$(*) \qquad \int_\Omega a_{ij}D_iuD_j\varphi \ge \int_\Omega f\varphi \quad \textit{for any } \varphi \in H^1_0(\Omega) \textit{ and } \varphi \ge 0 \textit{ in } \Omega.$$

*Suppose $f \in L^q(\Omega)$ for some $q > n/2$. Then for any $B_R \subset \Omega$ there holds for any $0 < p < n/(n-2)$ and any $0 < \theta < \tau < 1$*

$$\inf_{B_{\theta R}} u + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)} \ge C\left(\frac{1}{R^n}\int_{B_{\tau R}} u^p\right)^{\frac{1}{p}}$$

*where $C$ depends only on $n, p, q, \lambda, \Lambda, \theta$ and $\tau$.*

*Proof.* We prove for $R = 1$.

Step I. We prove that the result holds for some $p_0 > 0$.

Set $\bar{u} = u + k > 0$ for some $k > 0$ to be determined and $v = \bar{u}^{-1}$. First we will derive the equation for $v$. For any $\varphi \in H_0^1(B_1)$ with $\varphi \geq 0$ in $B_1$ consider $\bar{u}^{-2}\varphi$ as the test function in $(*)$. We have

$$\int_{B_1} a_{ij} D_i u \frac{D_j \varphi}{\bar{u}^2} - 2 \int_{B_1} a_{ij} D_i u D_j \bar{u} \frac{\varphi}{\bar{u}^3} \geq \int_{B_1} f \frac{\varphi}{\bar{u}^2}.$$

Note $D\bar{u} = Du$ and $Dv = -\bar{u}^2 D\bar{u}$. Therefore we obtain

$$\int_{B_1} a_{ij} D_j v D_i \varphi + \tilde{f} v \varphi \leq 0$$

where we set

$$\tilde{f} = \frac{f}{\bar{u}}.$$

In other words $v$ is a nonnegative subsolution to some homogeneous equation. Choose $k = \|f\|_{L^q}$ if $f$ is not identical zero. Otherwise choose arbitrary $k > 0$ and then let $k \to 0+$. Note

$$\|\tilde{f}\|_{L^q(B_1)} \leq 1.$$

Thus Theorem 1.1 implies that for any $\tau \in (\theta, 1)$ and any $p > 0$

$$\sup_{B_\theta} \bar{u}^{-p} \leq C \int_{B_\tau} \bar{u}^{-p}$$

i.e.,

$$\begin{aligned} \inf_{B_\theta} \bar{u} &\geq C \left( \int_{B_\tau} \bar{u}^{-p} dx \right)^{-\frac{1}{p}} \\ &= C \left( \int_{B_\tau} \bar{u}^{-p} \int_{B_\tau} \bar{u}^{p} \right)^{-\frac{1}{p}} \left( \int_{B_\tau} \bar{u}^{p} \right)^{\frac{1}{p}}, \end{aligned}$$

where $C = C(n, q, p, \lambda, \Lambda, \tau, \theta) > 0$.

The key point is to show that there exists a $p_0 > 0$ such that

$$\int_{B_\tau} \bar{u}^{-p_0} \cdot \int_{B_\tau} \bar{u}^{p_0} \leq C(n, q, \lambda, \Lambda, \tau).$$

We will show that for any $\tau < 1$ there holds

$$\int_{B_\tau} e^{p_0|w|} \le C(n, q, \lambda, \Lambda, \tau) \tag{1}$$

where $w = \log \bar{u} - \beta$ with $\beta = |B_\tau|^{-1} \int_{B_\tau} \log \bar{u}$.

We have two methods:

(i) Prove directly.

(ii) Prove that $w \in BMO$, i.e., for any $B_r(y) \subset B_1(0)$

$$\frac{1}{r^n} \int_{B_r} |w - w_{y,r}| dx \le C.$$

Then (1) follows from Theorem 1.5 in Chapter 3 (John-Nirenberg Lemma).

We shall prove (1) directly first. Recall $\bar{u} = u + k \ge k > 0$. Note that

$$e^{p_0|w|} = 1 + p_0|w| + \frac{(p_0|w|)^2}{2!} + \cdots + \frac{(p_0|w|)^n}{n!} + \cdots .$$

Hence we need to estimate

$$\int_{B_\tau} |w|^\beta$$

for each positive integer $\beta$.

We first derive the equation for $w$. Consider $\bar{u}^{-1}\varphi$ as test function in $(*)$. Here we need $\varphi \in L^\infty(B_1) \cap H_0^1(B_1)$ with $\varphi \ge 0$. By direct calculation as before and by the fact $Dw = \bar{u}^{-1} D\bar{u}$, we have

$$\begin{aligned} \int_{B_1} a_{ij} D_i w D_j w \varphi \le \int_{B_1} a_{ij} D_i w D_j \varphi + \int_{B_1} (-\tilde{f}\varphi) \\ \text{for any } \varphi \in L^\infty(B_1) \cap H_0^1(B_1) \text{ with } \varphi \ge 0. \end{aligned} \tag{2}$$

Replace $\varphi$ by $\varphi^2$ in (2). Hölder inequality implies

$$\int_{B_1} |Dw|^2 \varphi^2 \le C \Big\{ \int_{B_1} |D\varphi|^2 + \int_{B_1} |\tilde{f}|\varphi^2 \Big\}.$$

By Hölder inequality and Sobelev inequality we obtain

$$\int_{B_1} |\tilde{f}|\varphi^2 \le \|\tilde{f}\|_{L^{n/2}} \|\varphi\|^2_{L^{2n/(n-2)}} \le c(n, q) \|D\varphi\|^2_{L^2}.$$

Therefore we have

$$\int_{B_1} |Dw|^2\varphi^2 \le C\int_{B_1} |D\varphi|^2 \tag{3}$$

with $C = C(n, q, \lambda, \Lambda) > 0$. Take $\varphi \in C_0^1(B_1)$ with $\varphi \equiv 1$ in $B_\tau$. Then we obtain

$$\int_{B_\tau} |Dw|^2 \le C(n, q, \lambda, \Lambda, \tau). \tag{4}$$

Hence Poincaré inequality implies

$$\int_{B_\tau} w^2 \le c(n, \tau)\int_{B_\tau} |Dw|^2 \le C(n, q, \lambda, \Lambda, \tau)$$

since $\int_{B_\tau} w = 0$. Further more, we conclude from (3)

$$\int_{B_{\tau'}} w^2 \le C(n, q, \lambda, \Lambda, \tau, \tau') \tag{5}$$

for any $\tau' \in (\tau, 1)$.

Next we will estimate $\int_{B_\tau} |w|^\beta$ for any $\beta \ge 2$.

Choose $\varphi = \zeta^2|w_m|^{2\beta} \in H_0^1(B_1) \cap L^\infty(B_1)$ with $w_m = \begin{cases} -m & w \le -m \\ w & |w| < m \\ m & w \ge m \end{cases}$. Substitute such $\varphi$ in (2) to get

$$\begin{aligned}\int_{B_1} \zeta^2|w_m|^{2\beta}a_{ij}D_iwD_jw \le &(2\beta)\int_{B_1} \zeta^2 a_{ij}D_iwD_j|w_m||w_m|^{2\beta-1} \\ &+ \int_{B_1} 2\zeta|w_m|^{2\beta}a_{ij}D_iwD_j\zeta + \int_{B_1} |\tilde{f}|\zeta^2|w_m|^{2\beta}.\end{aligned}$$

Note $a_{ij}D_iwD_j|w_m| = a_{ij}D_iw_mD_j|w_m| \le a_{ij}D_iw_mD_jw_m$ a.e. in $B_1$. Young's inequality implies

$$\begin{aligned}(2\beta)|w_m|^{2\beta-1} &\le \frac{2\beta-1}{2\beta}|w_m|^{2\beta} + \frac{1}{2\beta}(2\beta)^{2\beta} \\ &= (1 - \frac{1}{2\beta})|w_m|^{2\beta} + (2\beta)^{2\beta-1}.\end{aligned}$$

Hence we obtain

$$\int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w D_j w \leq (1 - \frac{1}{2\beta}) \int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w_m D_j w_m$$
$$+ (2\beta)^{2\beta-1} \int_{B_1} \zeta^2 a_{ij} D_i w_m D_j w_m + \int_{B_1} 2\zeta |w_m|^{2\beta} a_{ij} D_i w D_j \zeta + \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta}$$

and hence

$$\int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w D_j w \leq (2\beta)^{2\beta} \int_{B_1} \zeta^2 a_{ij} D_i w_m D_j w_m$$
$$+ (4\beta) \int_{B_1} \zeta |w_m|^{2\beta} a_{ij} D_i w D_j \zeta + 2\beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta}.$$

Therefore we obtain

$$\int_{B_1} \zeta^2 |w_m|^{2\beta} |Dw|^2 \leq C\Big\{ (2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw_m|^2 + \beta \int_{B_1} \zeta |w_m|^{2\beta} |Dw||D\zeta|$$
$$+ \beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta} \Big\}.$$

Note the first term in the right side is bounded in (4). Applying Cauchy inequality to the second term in the right side we conclude

$$\int_{B_1} \zeta^2 |w_m|^{2\beta} |Dw|^2 \leq C\Big\{ (2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw_m|^2 + \beta^2 \int_{B_1} |w_m|^{2\beta} |D\zeta|^2$$
$$+ \beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta} \Big\}.$$

Note $Dw = Dw_m$ for $|w| < m$ and $Dw_m = 0$ for $|w| > m$. Hence we have

$$\int_{B_1} \zeta^2 |w_m|^{2\beta} |Dw_m|^2 \leq C\{ (2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw_m|^2 + \beta^2 \int_{B_1} |w_m|^{2\beta} |D\zeta|^2$$
$$+ \beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta} \}.$$

In the following, we write $w = w_m$ and then let $m \to +\infty$. By Young's inequality we obtain

$$|D(\zeta |w|^\beta)|^2 \leq 2|D\zeta|^2 |w|^{2\beta} + 2\beta^2 \zeta^2 |w|^{2\beta-2} |Dw|^2$$
$$\leq 2|D\zeta|^2 |w|^{2\beta} + 2\zeta^2 |Dw|^2 (\frac{\beta - 1}{\beta} |w|^{2\beta} + \frac{1}{\beta} \beta^{2\beta})$$

and hence

$$\int_{B_1} |D(\zeta|w|^\beta)|^2 \le C\Big\{(2\beta)^{2\beta}\int_{B_1}\zeta^2|Dw|^2+\beta^2\int|D\zeta|^2|w|^{2\beta} + \beta\int_{B_1}|\tilde f|\zeta^2|w|^{2\beta}\Big\}.$$

Hölder inequality implies

$$\int_{B_1}|\tilde f|\zeta^2|w|^{2\beta} \le \left(\int_{B_1}|\tilde f|^q\right)^{\frac{1}{q}}\left(\int_{B_1}(\zeta|w|^\beta)^{\frac{2q}{q-1}}\right)^{1-\frac{1}{q}}.$$

By interpolation inequality and Sobolev inequality with $2^*=\frac{2n}{n-2}>\frac{2q}{q-1}>2$ if $q>n/2$, we have

$$\begin{aligned}\|\zeta|w|^\beta\|_{L^{\frac{2q}{q-1}}} &\le \varepsilon\|\zeta|w|^\beta\|_{L^{2^*}}+C(n,q)\varepsilon^{-\frac{n}{2q-n}}\|\zeta|w|^\beta\|_{L^2}\\ &\le \varepsilon\|D(\zeta|w|^\beta)\|_{L^2}+C(n,q)\varepsilon^{-\frac{n}{2q-n}}\|\zeta|w|^\beta\|_{L^2}\end{aligned}$$

for any small $\varepsilon>0$. Therefore we obtain by (3)

$$\begin{aligned}\int_{B_1}|D(\zeta|w|^\beta)|^2 &\le C\Big\{(2\beta)^{2\beta}\int_{B_1}\zeta^2|Dw|^2+\beta^\alpha\int_{B_1}(|D\zeta|^2+\zeta^2)|w|^{2\beta}\Big\}\\ &\le C\Big\{(2\beta)^{2\beta}\int_{B_1}|D\zeta|^2+\beta^\alpha\int_{B_1}(|D\zeta|^2+\zeta^2)|w|^{2\beta}\Big\},\end{aligned}$$

for some positive constant $\alpha$ depending only on $n$ and $q$. Apply the Sobolev inequality for $\zeta|w|^\beta\in W_0^{1,2}(\mathbb{R}^n)$ with $\chi=\frac{n}{n-2}$ to get

$$\left(\int_{B_1}\zeta^{2\chi}|w|^{2\beta\chi}\right)^{\frac{1}{\chi}} \le C\beta^\alpha\Big\{(2\beta)^{2\beta}\int_{B_1}|D\zeta|^2+\int_{B_1}(|D\zeta|^2+\zeta^2)|w|^{2\beta}\Big\}.$$

Choose the cut-off function as follows. For $\tau\le r<R\le 1$, set $\zeta\equiv 1$ on $B_r(0)$, $\zeta\equiv 0$ in $B_1(0)\setminus B_R(0)$ and $|D\zeta|\le\frac{2}{R-r}$. Therefore we have

$$\left(\int_{B_r}|w|^{2\beta\chi}\right)^{\frac{1}{\chi}} \le \frac{C\beta^\alpha}{(R-r)^2}\{(2\beta)^{2\beta}+\int_{B_R}|w|^{2\beta}\}.$$

For some $\tau'\in(\tau,1)$ set $\beta_i=\chi^{i-1}$ and $r_i=\tau+\frac{1}{2^{i-1}}(\tau'-\tau)$ for any $i=1,2,\cdots$. Then for each $i=1,2,\cdots$,

$$\left(\int_{B_{r_i}}|w|^{2\chi^i}\right)^{\frac{1}{\chi}} \le \frac{C\chi^{(i-1)\alpha}2^{2(i-1)}}{(\tau'-\tau)^2}\Big\{(2\chi^{i-1})^{2\chi^{i-1}}+\int_{B_{r_{i-1}}}|w|^{2\chi^{i-1}}\Big\}.$$

Set

$$I_j = \|w\|_{L^{2\chi^j}(B_{r_j})}.$$

Then we have for $j = 1, 2, \cdots$,

$$I_j \le C^{\frac{j}{2\chi^j}} \{2\chi^{j-1} + I_{j-1}\}$$

with $C = C(n, q, \lambda, \Lambda, \tau, \tau') > 0$. Iterating the above inequality and observing that

$$\sum_{i=0}^{\infty} \frac{i}{\chi^i} < \infty,$$

we obtain

$$I_j \le C \sum_{i=1}^{j} \chi^{i-1} + C I_0,$$

i.e.,

$$I_j \le C\chi^j + C I_0.$$

Now for $\beta \ge 2$ there exists a $j$ such that $2\chi^{j-1} \le \beta < 2\chi^j$. Hence

$$\begin{aligned} I_\beta(B_\tau) \equiv \left( \int_{B_\tau} |w|^\beta \right)^{\frac{1}{\beta}} &\le C I_j \le C\chi^j + C I_0 \\ &\le C\beta + C I_0 \le C_0 \beta, \end{aligned}$$

since $I_0$ is bounded in (5). Hence we obtain for $\beta \ge 1$

$$\int_{B_\tau} |w|^\beta dx \le C_0^\beta \beta^\beta \le C_0^\beta e^\beta \beta!$$

where we used the Sterling formula for integer $\beta$. Hence for integer $\beta \ge 1$

$$\int_{B_\tau} \frac{(p_0|w|)^\beta}{\beta!} \le p_0^\beta (C_0 e)^\beta \le \frac{1}{2^\beta}$$

by choosing $p_0 = (2C_0 e)^{-1}$. This proves that

$$\begin{aligned} \int e^{p_0|w|} &= \int 1 + p_0|w| + \frac{(p_0|w|)^2}{2!} + \dots \\ &\le 1 + \frac{1}{2^1} + \frac{1}{2^2} + \dots \le 2. \end{aligned}$$

*Remark.* The above method, avoiding $BMO$, is elementrary in nature.

Now we give the second proof of the estimate (1). The estimate (3) gives

$$\int_{B_1} |Dw|^2\zeta^2 \le C\int_{B_1} |D\zeta|^2 \text{ for any } \zeta \in C_0^1(B_1).$$

Then for any $B_{2r}(y) \subset B_1$ choose $\zeta$ with

$$supp\ \zeta \subset B_{2r}(y),\ \zeta \equiv 1 \text{ in } B_r(y),\ |D\zeta| \le \frac{2}{r}.$$

Then we obtain

$$\int_{B_r(y)} |Dw|^2 \le Cr^{n-2}.$$

Hence Poincaré inequality implies

$$\begin{aligned}\frac{1}{r^n}\int_{B_r(y)} |w - w_{y,r}| &\le \frac{1}{r^{\frac{n}{2}}}\left(\int_{B_r(y)} |w - w_{y,r}|^2\right)^{\frac{1}{2}} \\ &\le \frac{1}{r^{\frac{n}{2}}}\left(r^2\int_{B_r(y)} |Dw|^2\right)^{\frac{1}{2}} \le C\end{aligned}$$

i.e., $w \in BMO$. Then John-Nirenberg Lemma implies

$$\int_{B_\tau} e^{p_0|w|} \le C.$$

Step II. The result holds for any positive $p < n/(n-2)$.

We need to prove for any $0 < r_1 < r_2 < 1$ and $0 < p_2 < p_1 < n/(n-2)$ there holds

$$\left(\int_{B_{r_1}} \bar{u}^{p_1}\right)^{\frac{1}{p_1}} \le C\left(\int_{B_{r_2}} \bar{u}^{p_2}\right)^{\frac{1}{p_2}} \tag{6}$$

for some $C = C(n, q, \lambda, \Lambda, r_1, r_2, p_1, p_2) > 0$.

Similar calculation may be found in section 3. Here we just point out some key steps.

Take $\varphi = \bar{u}^{-\beta}\eta^2$ for $\beta \in (0, 1)$ as the test function in $(*)$. Then we have

$$\int_{B_1} |D\bar{u}|^2\bar{u}^{-\beta-1}\eta^2 \le C\left\{\frac{1}{\beta^2}\int_{B_1} |D\eta|^2\bar{u}^{1-\beta} + \frac{1}{\beta}\int_{B_1} \frac{|f|}{k}\eta^2\bar{u}^{1-\beta}\right\}.$$

Set $\gamma = 1 - \beta \in (0, 1)$ and $w = \bar{u}^{\frac{\gamma}{2}}$. Then we have

$$\int |Dw|^2 \eta^2 \le \frac{C}{(1-\gamma)^\alpha} \int w^2 (|D\eta|^2 + \eta^2)$$

or

$$\int |D(w\eta)|^2 \le \frac{C}{(1-\gamma)^\alpha} \int w^2 (|D\eta|^2 + \eta^2),$$

for some positive $\alpha > 0$. Sobolev embedding theorem and appropriate choice of the cut-off function imply, with $\chi = n/n-2$, that for any $0 < r < R < 1$

$$\left( \int_{B_r} w^{2\chi} \right)^{\frac{1}{\chi}} \le \frac{C}{(1-\gamma)^\alpha} \cdot \frac{1}{(R-r)^2} \int_{B_R} w^2$$

or

$$\left( \int_{B_r} \bar{u}^{\gamma\chi} \right)^{\frac{1}{\gamma\chi}} \le \left( \frac{C}{(1-\gamma)^\alpha} \frac{1}{(R-r)^2} \right)^{\frac{1}{\gamma}} \left( \int_{B_R} \bar{u}^{\gamma} \right)^{\frac{1}{\gamma}}.$$

This holds for any $\gamma \in (0, 1)$. Now (6) follows after finitely many iterations.

Now the Harnack inequality is an easy consequence of above results.

**Theorem 3.3 (Moser's Harnack Inequality).** *Let $u \in H^1(\Omega)$ be a nonnegative solution in $\Omega$*

$$\int_\Omega a_{ij} D_i u D_j \varphi = \int_\Omega f\varphi \quad \text{for any } \varphi \in H_0^1(\Omega).$$

*Suppose $f \in L^q(\Omega)$ for some $q > n/2$. Then there holds for any $B_R \subset \Omega$,*

$$\max_{B_{\frac{R}{2}}} u \le C \left\{ \min_{B_{\frac{R}{2}}} u + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \right\}$$

*where $C = C(n, \lambda, \Lambda, q)$ is a positive constant.*

**Corollary 3.4 (Hölder Continuity).** *Let $u \in H^1(\Omega)$ be a solution in $\Omega$*

$$\int_\Omega a_{ij} D_i u D_j \varphi = \int_\Omega f\varphi \quad \text{for any } \varphi \in H_0^1(\Omega).$$

*Suppose $f \in L^q(\Omega)$ for some $q > n/2$. Then $u \in C^\alpha(\Omega)$ for some $\alpha \in (0, 1)$ depending only on $n, q, \lambda$ and $\Lambda$. Moreover there holds for any $B_R \subset \Omega$,*

$$|u(x) - u(y)| \le C \left( \frac{|x-y|}{R} \right)^\alpha \left\{ \left( \frac{1}{R^n} \int_{B_R} u^2 \right)^{\frac{1}{2}} + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \right\}$$

$$\text{for any } x, y \in B_{\frac{R}{2}}$$

*where $C = C(n, \lambda, \Lambda, q)$ is a positive constant.*

*Proof.* We prove the estimate for $R = 1$. Let $M(r) = \max_{B_r} u$ and $m(r) = \min_{B_r} u$ for $r \in (0, 1)$. Then $M(r) < +\infty$ and $m(r) > -\infty$. It suffices to show that

$$\omega(r) \triangleq M(r) - m(r) \le Cr^\alpha \left\{ (\int_{B_1} u^2)^{\frac{1}{2}} + \|f\|_{L^q(B_1)} \right\} \text{ for any } r < \frac{1}{2}.$$

Set $\delta = 2 - \frac{n}{q}$. Apply the Theorem 3.3 to $M(r) - u \ge 0$ in $B_r$ to get

$$\sup_{B_{\frac{r}{2}}}(M(r) - u) \le C \left\{ \inf_{B_{\frac{r}{2}}}(M(r) - u) + r^\delta \|f\|_{L^q(B_r)} \right\},$$

i.e.,

$$M(r) - m(\frac{r}{2}) \le C \left\{ (M(r) - M(\frac{r}{2})) + r^\delta \|f\|_{L^q(B_r)} \right\}. \tag{1}$$

Similarly, apply Harnack to $u - m(r) \ge 0$ in $B_r$ to get

$$M(\frac{r}{2}) - m(r) \le C \left\{ (m(\frac{r}{2}) - m(r)) + r^\delta \|f\|_{L^q(B_r)} \right\}. \tag{2}$$

Then by adding (1) and (2) together we get

$$\omega(r) + \omega(\frac{r}{2}) \le C \left\{ (\omega(r) - \omega(\frac{r}{2})) + r^\delta \|f\|_{L^q(B_r)} \right\}$$

or

$$\omega(\frac{r}{2}) \le \gamma\omega(r) + Cr^\delta \|f\|_{L^q(B_r)}$$

for some $\gamma = \frac{C-1}{C+1} < 1$.

Apply Lemma 3.5 below with $\mu$ chosen such that $\alpha = (1 - \mu) \log \gamma / \log \tau < \mu\delta$. We obtain

$$\omega(\rho) \le C\rho^\alpha \left\{ \omega(\frac{1}{2}) + \|f\|_{L^q(B_1)} \right\} \quad \text{for any } \rho \in (0, \frac{1}{2}].$$

While Theorem 3.1 implies

$$\omega(\frac{1}{2}) \le C \left\{ \left( \int_{B_1} u^2 \right)^{\frac{1}{2}} + \|f\|_{L^q(B_1)} \right\}.$$

**Lemma 3.5.** *Let $\omega$ and $\sigma$ be non-decreasing functions in an interval $(0, R]$. Suppose there holds for all $r \le R$*

$$\omega(\tau r) \le \gamma\omega(r) + \sigma(r)$$

*for some $0 < \gamma, \tau < 1$. Then for any $\mu \in (0,1)$ and $r \le R$ we have*

$$\omega(r) \le C\left\{ (\frac{r}{R})^{\alpha}\omega(R) + \sigma(r^{\mu}R^{1-\mu}) \right\}$$

*where $C = C(\gamma, \tau)$ and $\alpha = \alpha(\gamma, \tau, \mu)$ are positive constants. In fact $\alpha = (1 - \mu)\log\gamma/\log\tau$.*

*Proof.* Fix some number $r_1 \le R$. Then for any $r \le r_1$ we have

$$\omega(\tau r) \le \gamma\omega(r) + \sigma(r_1)$$

since $\sigma$ is nondecreasing. We now iterate this inequality to get for any positive integer $k$

$$\omega(\tau^k r_1) \le \gamma^k\omega(r_1) + \sigma(r_1)\sum_{i=0}^{k-1}\gamma^i \le \gamma^k\omega(R) + \frac{\sigma(r_1)}{1-\gamma}.$$

For any $r \le r_1$ we choose $k$ in such a way that

$$\tau^k r_1 < r \le \tau^{k-1} r_1.$$

Hence we have

$$\begin{aligned}\omega(r) &\le \omega(\tau^{k-1}r_1) \le \gamma^{k-1}\omega(R) + \frac{\sigma(r_1)}{1-\gamma} \\ &\le \frac{1}{\gamma}(\frac{r}{r_1})^{\log\gamma/\log\tau}\omega(R) + \frac{\sigma(r_1)}{1-\gamma}.\end{aligned}$$

Now let $r_1 = r^{\mu}R^{1-\mu}$. We obtain

$$\omega(r) \le \frac{1}{\gamma}(\frac{r}{R})^{(1-\mu)(\log\gamma/\log\tau)}\omega(R) + \frac{\sigma(r^{\mu}R^{1-\mu})}{1-\gamma}.$$

This finishes the proof.

**Corollary 3.6 (Liouville Theorem).** *Suppose $u$ is a solution to a homogeneous equation in $\mathbb{R}^n$*

$$\int_{\mathbb{R}^n} a_{ij} D_i u D_j \varphi = 0 \quad \textit{for any } \varphi \in H_0^1(\mathbb{R}^n).$$

*If $u$ is bounded, then $u$ is a constant.*

*Proof.* We showed that there exists a $\gamma < 1$ such that

$$\omega(r) \le \gamma \omega(2r).$$

By iteration we have

$$\omega(r) \le \gamma^k \omega(2^k r) \to 0 \text{ as } k \to \infty$$

since $\omega(2^k r) \le C$ if $u$ is bounded. Hence for any $r > 0$

$$\omega(r) = 0.$$

## §4. Nonlinear Equations

Up to now, we have been discussing linear equations of the following form

$$-D_j(a_{ij}(x) D_i u) = f(x) \text{ in } B_1.$$

It is natural to ask how they generalize to nonlinear equations. To answer this question, let us consider equation for a solution $v$ with the form

$$v(x) = \Phi(u(x))$$

for some smooth function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi' \ne 0$. Any estimates for $u$ can be translated to those for $v$. To find the equation for $v$, we write

$$u = \Psi(v)$$

with $\Psi = \Phi^{-1}$. Then by setting $\eta = \Psi'(v)\xi$ for $\xi \in C_0^\infty(B_1)$ we have

$$\begin{aligned}\int a_{ij} D_i u D_j \xi &= \int a_{ij} \Psi'(v) D_i v D_j \xi \\ &= \int a_{ij} D_i v D_j \eta - \int \frac{\Psi''(v)}{\Psi'(v)} a_{ij} D_i v D_j v \eta.\end{aligned}$$

Therefore if $u$ is a solution

$$\int a_{ij} D_i u D_j \xi = \int f(x) \xi \quad \text{for any} \quad \xi \in H_0^1(B_1)$$

then $v$ satisfies

$$\int a_{ij} D_i v D_j \eta = \int \left( \frac{\Psi''(v)}{\Psi'(v)} a_{ij} D_i v D_j v + \frac{1}{\Psi'(v)} f \right) \eta \quad \text{for any} \quad \eta \in C_0^\infty(B_1).$$

Note that the nonlinear term has quadratic growth in terms of $Dv$. Hence we may extend the space of test functions to $H_0^1(B_1) \cap L^\infty(B_1)$. It turns out that $H^1(B_1) \cap L^\infty(B_1)$ is also the right space for the solution. The following example illustrates the boundedness of solutions is essential.

*Example.* Consider the equation

$$-\Delta u = |Du|^2$$

in the ball $B_R(0)$ in $\mathbb{R}^2$ with $R < 1$. It is easy to check that $u(x) = \log\log |x|^{-1} - \log\log R^{-1} \in H^1(B_R(0))$ is a weak solution with zero boundary data. Note that $u(x) \equiv 0$ is also a solution.

In this section, we always assume $a_{ij} \in L^\infty(B_1)$ satisfies

$$\lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \text{ for any } x \in B_1 \text{ and } \xi \in \mathbb{R}^n$$

for some positive constants $\lambda$ and $\Lambda$. We consider the nonlinear equation of the form

$$(*) \qquad \int a_{ij}(x) D_i u D_j \varphi = \int b(x, u, Du) \varphi \text{ for any } \varphi \in H_0^1(B_1) \cap L^\infty(B_1).$$

We say the nonlinear term $b$ satisfies the natural growth condition if

$$|b(x, u, p)| \leq C(u)(f(x) + |p|^2) \text{ for any } (x, u, p) \in B_1 \times \mathbb{R} \times \mathbb{R}^n$$

for some constant $C(u)$ depending only on $u$ and $f \in L^q(B_1)$ for some $q \geq \frac{2n}{n+2}$. We always assume

$$u \in H^1(B_1) \cap L^\infty(B_1).$$

**Lemma 4.1.** *Suppose $u \in H^1(B_1)$ is a nonnegative solution of $(*)$ with $|u| \leq M$ in $B_1$ and that $b$ satisfies the natural growth condition with $f(x) \in L^q(B_1)$ for some $q > \frac{n}{2}$. Then for any $B_R \subset B_1$ there holds*

$$\sup_{B_{\frac{R}{2}}} u \leq C\left\{\inf_{B_{\frac{R}{2}}} u + R^{2-\frac{n}{q}}\left(\int_{B_R} |f|^q\right)^{\frac{1}{q}}\right\}$$

*where $C$ is a positive constant depending only on $n, \lambda, \Lambda, M$ and $q$.*

*Proof.* Let $v = \frac{1}{\alpha}(e^{\alpha u} - 1)$ for some $\alpha > 0$. Then for $\varphi \in H_0^1(B_1) \cap L^\infty(B_1)$ with $\varphi \geq 0$ there holds

$$\begin{aligned}
\int a_{ij} D_i v D_j \varphi &= \int a_{ij} e^{\alpha u} D_i u D_j \varphi \\
&= \int a_{ij} D_i u D_j (e^{\alpha u}\varphi) - \alpha \int a_{ij} e^{\alpha u} D_i u D_j u \varphi \\
&= \int b(x, u, Du) e^{\alpha u}\varphi - \alpha \int a_{ij} e^{\alpha u} D_i u D_j u \varphi \\
&\leq C(M) \int (f(x) + |Du|^2) e^{\alpha u}\varphi - \alpha\lambda \int |Du|^2 e^{\alpha u}\varphi.
\end{aligned}$$

Hence by taking $\alpha$ large we have

$$\int a_{ij} D_i v D_j \varphi \leq C \int f(x)\varphi \text{ for any } \varphi \in H_0^1(B_1) \cap L^\infty(B_1) \text{ with } \varphi \geq 0, \tag{1}$$

for some positive constant $C$ depending only on $n, \lambda, \Lambda$ and $M$. Observe that $u$ and $v$ are compatible. Therefore by Theorem 3.1 we obtain for any $p > 0$

$$\begin{aligned}
\sup_{B_{\frac{R}{2}}} u &\leq C(M, \alpha) \sup_{B_{\frac{R}{2}}} v \\
&\leq C\left\{\left(\frac{1}{R^n}\int_{B_R} v^p\right)^{\frac{1}{p}} + R^{2-\frac{n}{q}}\left(\int_{B_R} f^q\right)^{\frac{1}{q}}\right\} \\
&\leq C\left\{\left(\frac{1}{R^n}\int_{B_R} u^p\right)^{\frac{1}{p}} + R^{2-\frac{n}{q}}\left(\int_{B_R} f^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

For the lower bound, we let $w = \frac{1}{\alpha}(1 - e^{-\alpha u})$. As before by choosing $\alpha > 0$ large we have

$$\int a_{ij} D_i w D_j \varphi \geq C \int f(x)\varphi \text{ for any } \varphi \in H_0^1(B_1) \cap L^\infty(B_1) \text{ with } \varphi \geq 0.$$

Hence by Theorem 3.2, we obtain for any $p \in (0, \frac{n}{n-2})$

$$\left(\frac{1}{R^n}\int_{B_R} u^p\right)^{\frac{1}{p}} \le C\left\{\inf_{B_{\frac{R}{2}}} u + R^{2-\frac{n}{q}}\left(\int_{B_R} f^q\right)^{\frac{1}{q}}\right\}.$$

Combining the above inequalities we prove the Lemma 4.1.

*Remark.* In estimate (1) in the above proof, take $\varphi = (u+M)\eta^2$ for some $\eta \in C_0^1(B_1)$. Then by Hölder inequality we conclude

$$\int |Du|^2\eta^2 \le C\left\{\int \left(|D\eta|^2 + |f|\eta^2\right)\right\}$$

for some positive constant $C$ depending only on $n, \lambda, \Lambda$ and $M$. This implies the interior $L^2$-estimate of gradient $Du$ in terms of these constants together with $\|f\|_{L^1(B_1)}$. This fact will be used in the proof of Theorem 4.3.

**Corollary 4.2.** *Suppose $u \in H^1(B_1)$ is a bounded solution of $(*)$ and that $b$ satisfies the natural growth condition with $f(x) \in L^q(B_1)$ for some $q > \frac{n}{2}$. Then $u \in C^\alpha_{loc}(B_1)$ with $\alpha = \alpha(n, \lambda, \Lambda, q, |u|_{L^\infty})$. Moreover there holds*

$$|u(x) - u(y)| \le C|x-y|^\alpha \quad \text{for any } x, y \in B_{\frac{1}{2}}$$

*where $C$ is a positive constant depending only on $n, \lambda, \Lambda, q$, $|u|_{L^\infty(B_1)}$ and $\|f\|_{L^q(B_1)}$.*

*Proof.* The proof is identical to that of Theorem 3.4, with Theorem 3.3 replaced by Lemma 4.1.

**Theorem 4.3.** *Suppose $u \in H^1(B_1)$ is a bounded solution of $(*)$ and that $b$ satisfies the natural growth condition with $f \in L^q(B_1)$ for some $q > n$. Assume further that $a_{ij} \in C^\alpha(B_1)$ for $\alpha = 1 - \frac{n}{q}$. Then $Du \in C^\alpha_{loc}(B_1)$. Moreover there holds*

$$|Du|_{C^\alpha(B_{\frac{1}{2}})} \le C(n, \lambda, \Lambda, q, |u|_{L^\infty(B_1)}, \|f\|_{L^q(B_1)}).$$

*Proof.* We only need to prove $Du \in L^\infty_{loc}$. Then the Hölder continuity is implied by Theorem 3.1 Chapter 3. For any $B_r(x_0) \subset B_1$ solve for $w$ such that

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i w D_j \varphi = 0 \text{ for any } \varphi \in H_0^1(B_r(x_0))$$

with $w - u \in H_0^1(B_r(x_0))$. Then the maximum principle implies

$$\inf_{B_r(x_0)} u \le w \le \sup_{B_r(x_0)} u \quad \text{in } B_r(x_0)$$

or

$$\sup_{B_r(x_0)} |u-w| \le \underset{B_r(x_0)}{osc}\, u. \tag{1}$$

By Lemma 2.2 in Chapter 3, we have for any $0 < \rho \le r$,

$$\int_{B_\rho(x_0)} |Du|^2 \le c\left\{\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Du|^2 + \int_{B_r(x_0)} |D(u-w)|^2\right\} \tag{2}$$

and

$$\begin{aligned}\int_{B_\rho(x_0)} |Du-(Du)_{x_0,\rho}|^2 \le c\Big\{ \left(\frac{\rho}{r}\right)^{n+2} &\int_{B_r(x_0)} |Du-(Du)_{x_0,r}|^2 \\ &+ \int_{B_r(x_0)} |D(u-w)|^2\Big\}.\end{aligned} \tag{3}$$

Note that the function $v = u - w \in H_0^1(B_r(x_0))$ satisfies

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i v D_j \varphi = \int_{B_r(x_0)} b(x,u,Du)\varphi + \int_{B_r(x_0)} \big(a_{ij}(x_0) - a_{ij}(x)\big) D_i u D_j \varphi$$
$$\varphi \in H_0^1(B_r(x_0)) \cap L^\infty(B_r(x_0)).$$

Taking $\varphi = v$ and by Sobolev inequality we obtain

$$\int_{B_r(x_0)} |Dv|^2 \le C\Big\{ \int_{B_r(x_0)} |Du|^2|v| + r^{2\alpha}\int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha}\|f\|^2_{L^q(B_1)}\Big\}.$$

Hence with (1) we conclude

$$\int_{B_r(x_0)} |Dv|^2 \le C\Big\{\Big(r^{2\alpha} + \underset{B_r(x_0)}{osc}\, u\Big)\int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha}\|f\|^2_{L^q}\Big\}. \tag{4}$$

Corollary 4.2 implies $u \in C^{\delta_0}$ for some $\delta_0 > 0$. Therefore we have by (2) and (4)

$$\int_{B_\rho(x_0)} |Du|^2 \le C\left\{\left[\left(\frac{\rho}{r}\right)^n + r^{2\alpha} + r^{\delta_0}\right]\int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha}\|f\|^2_{L^q}\right\}.$$

By Lemma 1.4 in Chapter 3 we obtain for any $\delta < 1$ there holds for any $B_r(x_0) \subset B_{\frac{7}{8}}$

$$\int_{B_r(x_0)} |Du|^2 \le Cr^{n-2+2\delta}\Big\{\int_{B_{\frac{7}{8}}} |Du|^2 + \|f\|^2_{L^q(B_1)}\Big\}.$$

This implies $u \in C^{\delta}_{\text{loc}}$ for any $\delta < 1$. Moreover for any $B_r(x_0) \subset B_{\frac{3}{4}}$ there holds

$$\underset{B_r(x_0)}{osc}\ u \le Cr^{\delta}$$

for some positive constant depending only on $n, \lambda, \Lambda, q$, $|u|_{L^\infty(B_1)}$ and $\|f\|_{L^q(B_1)}$, by the remark after the proof of Lemma 4.1. With (4) we have for any $B_r(x_0) \subset B_{\frac{2}{3}}$

$$\begin{aligned}\int_{B_r(x_0)} |Dv|^2 \le& C\left\{(r^{2\alpha} + r^{\delta})r^{n-2+2\delta} \int_{B_{\frac{7}{8}}} |Du|^2 + r^{n+2\alpha}\|f\|^2_{L^q}\right\}\\ \le& Cr^{n+2\alpha'}\end{aligned}$$

for some $\alpha' < \alpha$ if $\delta \in (0,1)$ is chosen such that $3\delta > 2$ and $\alpha + \delta > 1$. Hence with (3) we obtain for any $B_r(x_0) \subset B_{\frac{2}{3}}$ and any $0 < \rho \le r$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\left\{ \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + r^{n+2\alpha'} \right\}.$$

By Lemma 1.4 and Theorem 1.1 in Chapter 3 again we conclude that $Du \in C^{\alpha'}_{loc}$ for some $\alpha' < \alpha$, in particular $Du \in L^{\infty}_{loc}$. This finishes the proof.

## CHAPTER 5
## VISCOSITY SOLUTIONS

### GUIDE

Here we only try to explain a few basic ides in obtaining estimates for viscosity solutions.Students should read the book for further developments.

In this chapter we generalize the notion of classical solutions to viscosity solutions and study their regularities. We define viscosity solutions by comparing them with quadratic polynomials and thus remove the requirement that solutions be at least $C^2$. The main tool to study viscosity solutions is the maximum principle due to Alexandroff. We first generalize such maximum principle to viscosity solutions and then use the resulting estimate to discuss the regularity theory. We use it to control the distribution functions of solutions and obtain Harnack inequality, and hence $C^\alpha$ regularity, which generalize a result by Krylov and Safonov. We also use it to approximate solutions in $L^\infty$ by quadratic polynomials and get Schauder ($C^{2,\alpha}$) estimates. The methods are basically nonlinear, in the sense that they do not rely on differentiating equations. This implies that the results obtained in this way may apply to general fully nonlinear equations, although in this note we focus only on linear equations.

### §1. Alexandroff Maximum Principle

We begin this section with the definition of viscosity solutions. This very weak concept of solutions enables us to define a class of functions containing all classical solutions of linear and nonlinear elliptic equations with fixed ellipticity constants and bounded measurable coefficients.

Suppose that $\Omega$ is a bounded and connected domain in $\mathbb{R}^n$ and that $a_{ij} \in C(\Omega)$ satisfies

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \text{ for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n$$

for some positive constants $\lambda$ and $\Lambda$. Consider the operator $L$ in $\Omega$ defined by

$$Lu \equiv a_{ij}(x)D_{ij}u$$

for $u \in C^2(\Omega)$.

Suppose $u \in C^2(\Omega)$ is a supersolution in $\Omega$, i.e., $Lu \le 0$. Then for any $\varphi \in C^2(\Omega)$ with $L\varphi > 0$ we have

$$L(u - \varphi) < 0 \text{ in } \Omega.$$

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This implies by the maximum principle that $u-\varphi$ cannot have local interior minimums in $\Omega$. In other words if $u-\varphi$ has a local minimum at $x_0 \in \Omega$, there holds

$$L\varphi(x_0) \leq 0.$$

Geometrically $u-\varphi$ having a local minimum at $x_0$ means that $\varphi$ touches $u$ from below at $x_0$ if we adjust $\varphi$ appropriately by adding a constant. This suggests the following definition. We assume $f \in C(\Omega)$.

**Definition.** $u \in C(\Omega)$ is a viscosity supersolution (resp. subsolution) of

$$Lu = f \text{ in } \Omega$$

if for any $x_0 \in \Omega$ and any function $\varphi \in C^2(\Omega)$ such that $u-\varphi$ has a local minimum (resp. maximum) at $x_0$ there holds

$$L\varphi(x_0) \leq f(x_0) \qquad (\text{resp. } L\varphi(x_0) \geq f(x_0)).$$

We say that $u$ is a viscosity solution if it is a viscosity subsolution and a viscosity supersolution.

*Remark.* By approximation we may replace the $C^2$ function $\varphi$ by a quadratic polynomial $Q$.

*Remark.* The above analysis shows that a classical supersolution is a viscosity supersolution. It is straightforward to prove that a $C^2$ viscosity supersolution is a classical supersolution. Similar statements hold for subsolutions and solutions.

*Remark.* The notion of viscosity solutions can be generalized to nonlinear equations accordingly.

Now we define in a weak way the class of "all solutions to all elliptic equations". For any function $\varphi$, which is $C^2$ at $x_0$, we have the following equivalence

$$\begin{aligned}
&\sum_{i,j=1}^{n} a_{ij}(x_0) D_{ij}\varphi(x_0) \leq 0 \\
\Longleftrightarrow &\sum_{k=1}^{n} \alpha_k e_k \leq 0 \text{ with } \lambda \leq \alpha_k \leq \Lambda, e_k = e_k(D^2\varphi(x_0)) \\
\Longleftrightarrow &\sum_{e_i>0} \alpha_i e_i + \sum_{e_i<0} \alpha_i e_i \leq 0 \\
\Longleftrightarrow &\sum_{e_i>0} \alpha_i e_i \leq \sum_{e_i<0} \alpha_i(-e_i),
\end{aligned}$$

which implies

$$\lambda \sum_{e_i>0} e_i \le \Lambda \sum_{e_i<0} (-e_i)$$

where $e_1, \cdots, e_n$ are eigenvalues of the Hessian matrix $D^2\varphi(x_0)$. This means that positive eigenvalues of $D^2\varphi(x_0)$ are controlled by negative eigenvalues.

**Definition.** Suppose $f$ is a continuous function in $\Omega$ and that $\lambda$ and $\Lambda$ are two positive constants. We define $u \in C(\Omega)$ to belong to $\mathcal{S}^+(\lambda, \Lambda, f)$ (resp. $\mathcal{S}^-(\lambda, \Lambda, f)$) if for any $x_0 \in \Omega$ and any function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum (resp. maximum) at $x_0$ there holds

$$\lambda \sum_{e_i>0} e_i(x_0) + \Lambda \sum_{e_i<0} e_i(x_0) \le f(x_0)$$
$$(\text{resp.} \quad \Lambda \sum_{e_i>0} e_i(x_0) + \lambda \sum_{e_i<0} e_i(x_0) \ge f(x_0))$$

where $e_1(x_0), \cdots, e_n(x_0)$ are eigenvalues of the Hessian matrix $D^2\varphi(x_0)$.

We denote $\mathcal{S}(\lambda, \Lambda, f) = \mathcal{S}^+(\lambda, \Lambda, f) \cap \mathcal{S}^-(\lambda, \Lambda, f)$.

*Remark.* Any viscosity supersolutions of

$$a_{ij}D_{ij}u = f \text{ in } \Omega$$

belong to the class $\mathcal{S}^+(\lambda, \Lambda, f)$ where there holds

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \text{ for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n.$$

The class $\mathcal{S}^+(\lambda, \Lambda, f)$ and $\mathcal{S}^-(\lambda, \Lambda, f)$ also include solutions to fully nonlinear equations. Among them are the Pucci's equations.

*Example.* For any two positive constants $\lambda \le \Lambda$ let $A$ be a symmetric matrix whose eigenvalues belong to $[\lambda, \Lambda]$, i.e., $\lambda|\xi|^2 \le A_{ij}\xi_i\xi_j \le \Lambda|\xi|^2$ for any $\xi \in \mathbb{R}^n$. Let $\mathcal{A}_{\lambda,\Lambda}$ denote the class of all such matrices. For any symmetric matrix $M$ we define the Pucci's extremal operators

$$\mathcal{M}^-(M) = \mathcal{M}^-(\lambda, \Lambda, M) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} A_{ij}M_{ij}$$
$$\mathcal{M}^+(M) = \mathcal{M}^+(\lambda, \Lambda, M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} A_{ij}M_{ij}.$$

Pucci's equations are given by

$$\mathcal{M}^-(\lambda, \Lambda, M) = f, \quad \mathcal{M}^+(\lambda, \Lambda, M) = g$$

for continuous functions $f$ and $g$ in $\Omega$. It is easy to see that

$$\mathcal{M}^-(\lambda, \Lambda, M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i$$
$$\mathcal{M}^+(\lambda, \Lambda, M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i$$

where $e_1, \cdots, e_n$ are eigenvalues of $M$. Therefore $u \in \mathcal{S}^+(\lambda, \Lambda, f)$ if and only if $\mathcal{M}^-(\lambda, \Lambda, D^2 u) \le f$ in the viscosity sense, i.e., for any $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at $x_0 \in \Omega$ there holds

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0)) \le f(x_0).$$

By the definition of $\mathcal{M}^-$ and $\mathcal{M}^+$ it is easy to check that for any two symmetric matrices $M$ and $N$

$$\begin{aligned}\mathcal{M}^-(M) + \mathcal{M}^-(N) \le \mathcal{M}^-(M+N) &\le \mathcal{M}^+(M) + \mathcal{M}^-(N)\\ &\le \mathcal{M}^+(M+N) \le \mathcal{M}^+(M) + \mathcal{M}^+(N).\end{aligned}$$

This property will be needed in Section 2.

Next we derive the Alexandroff maximum principle for viscosity solutions. It replaces the energy inequality for solutions to equations of divergence forms.

Let $v$ be a continuous function in an open convex set $\Omega$. Recall that the convex envelope of $v$ in $\Omega$ is defined by

$$\Gamma(v)(x) = \sup_L \{L(x); L \le v \text{ in } \Omega, L \text{ is an affine function}\}$$

for any $x \in \Omega$. It is easy to see that $\Gamma(v)$ is a convex function in $\Omega$. The set $\{v = \Gamma(v)\} = \{x \in \Omega; v(x) = \Gamma(v)(x)\}$ is called the (lower) contact set of $v$. The points in the contact set are called contact points.

The following is the classical version of the Alexandroff maximum principle. We do not require that functions be solutions to elliptic equations. See Lemma 4.2 in Chapter 2.

**Lemma 1.1.** *Suppose $u$ is a $C^{1,1}$ function in $B_1$ with $u \ge 0$ on $\partial B_1$. Then there holds*

$$\sup_{B_1} u^- \le c(n) \left( \int_{B_1 \cap \{u = \Gamma_u\}} det D^2 u \right)^{\frac{1}{n}}$$

*where $\Gamma_u$ is the convex envelope of $-u^- = \min\{u, 0\}$.*

Now we state the viscosity version.

**Theorem 1.2.** *Suppose $u$ belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_1$ with $u \geq 0$ on $\partial B_1$ for some $f \in C(\Omega)$. Then there holds*

$$\sup_{B_1} u^- \leq c(n, \lambda, \Lambda)\Big(\int_{B_1 \cap \{u = \Gamma_u\}} (f^+)^n\Big)^{\frac{1}{n}}$$

*where $\Gamma_u$ is the convex envelope of $-u^- = \min\{u, 0\}$..*

*Proof.* We will prove that $\Gamma_u$ is a $C^{1,1}$ function in $B_1$ and that at contact point $x_0$ there hold

$$f(x_0) \geq 0 \tag{1}$$

and

$$L(x) \leq \Gamma_u(x) \leq L(x) + C\big\{f(x_0) + \varepsilon(x)\big\}|x - x_0|^2 \tag{2}$$

for some affine function $L$ and any $x$ close to $x_0$, where $\varepsilon(x) \to 0$ as $x \to x_0$ and $C$ is a positive constant depending only on $n, \lambda$ and $\Lambda$. We obtain by (2)

$$\det D^2\Gamma_u(x) \leq C(n, \lambda, \Lambda)\big(f(x)\big)^n \quad \text{for a.e. } x \in \{u = \Gamma_u\}.$$

We may apply Lemma 1.1 to function $\Gamma_u$ to get the result.

Suppose $x_0$ is a contact point, i.e., $u(x_0) = \Gamma_u(x_0)$. We may assume $x_0 = 0$. We also assume, by subtracting a supporting plane at $x_0 = 0$, that $u \geq 0$ in $B_1$ and that $u(0) = 0$.

In order to prove (1) we take $h(x) = -\varepsilon|x|^2/2$ in $B_1$. Obviously that $u - h$ has a minimum at 0. Note that the eigenvalues of $D^2h(0)$ are $-\varepsilon, \cdots, -\varepsilon$. By definition of $\mathcal{S}^+(\lambda, \Lambda, f)$ we have

$$-n\Lambda\varepsilon \leq f(0).$$

By letting $\varepsilon \to 0$ we get (1).

For estimate (2) we will prove

$$0 \leq \Gamma_u(x) \leq C(n, \lambda, \Lambda)\big\{f(0) + \varepsilon(x)\big\}|x|^2 \text{ for } x \in B_1$$

where $\varepsilon(x) \to 0$ as $x \to 0$. By setting $w = \Gamma_u$ we need to estimate for any small $r > 0$

$$C_r = \frac{1}{r^2}\max_{B_r} w.$$

Fix $r > 0$. By convexity $w$ attains its maximum in $\bar{B}_r$ at some point on the boundary, say, $(0, \cdots, 0, r)$. The set $\{x \in B_1; w(x) \le w(0, \cdots, 0, r)\}$ is convex and contains $B_r$. It follows easily that

$$w(x', r) \ge w(0, \cdots, 0, r) = C_r r^2 \text{ for any } x = (x', r) \in B_1.$$

Take a positive number $N$ to be determined. Set

$$R_r = \{(x', x_n); |x'| \le Nr, |x_n| \le r\}.$$

We will construct a quadratic polynomial that touches $u$ from below in $R_r$ and curves upward very much. Set for some $b > 0$

$$h(x) = (x_n + r)^2 - b|x'|^2.$$

Then we have

(i) for $x_n = -r$, $h \le 0$;

(ii) for $|x'| = Nr$, $h \le (4 - bN^2)r^2 \le 0$ if we take $b = 4/N^2$;

(iii) for $x_n = r$, $h = 4r^2 - b|x'|^2 \le 4r^2$.

Hence if we set

$$\tilde{h}(x) = \frac{C_r}{4} h(x) = \frac{C_r}{4}\{(x_n + r)^2 - \frac{4}{N^2}|x'|^2\}$$

we obtain $\tilde{h} \le w \le u$ on $\partial R_r$ (since $w$ is the convex envelope of $u$) and $\tilde{h}(0) = C_r r^2/4 > 0 = w(0) = u(0)$. By lowering $\tilde{h}$ appropriately we conclude that $u - \tilde{h}$ has a local minimum somewhere inside $R_r$. Note the eigenvalues of $D^2\tilde{h}$ are given by $C_r/2, -2C_r/N^2, \cdots, -2C_r/N^2$. Hence by definition of $\mathcal{S}^+(\lambda, \Lambda, f)$ we have

$$\lambda \frac{C_r}{2} - 2\Lambda(n-1)\frac{C_r}{N^2} \le \max_{R_r} f.$$

By choosing $N$ large, depending only on $n, \lambda$ and $\Lambda$, we obtain

$$C_r \le \frac{4}{\lambda} \max_{R_r} f$$

or

$$\max_{B_r} w \le \frac{4}{\lambda} r^2 \max_{R_r} f.$$

Note $\max_{R_r} f \to f(0)$ as $r \to 0$. This finishes the proof.

We end this section with a simple consequence of Calderon-Zygmund decomposition. We first recall some terminology. Let $Q_1$ be the unit cube. Cut it equally into $2^n$ cubes, which we take as the first generation. Do the same cutting for these small cubes to get the second generation. Continue this process. These cubes (from all generations) are called *dyadic cubes*. Any $(k+1)$-generation cube $Q$ comes from some $k$-generation cube $\tilde{Q}$, which is called the *predecessor* of $Q$.

**Lemma 1.3.** *Suppose measurable sets $A \subset B \subset Q_1$ have the following properties*

*(i) $|A| < \delta$ for some $\delta \in (0, 1)$;*

*(ii) for any dyadic cube $Q$, $|A \cap Q| \geq \delta |Q|$ implies $\tilde{Q} \subset B$ for the predecessor $\tilde{Q}$ of $Q$.*

*Then there holds*

$$|A| \leq \delta |B|.$$

*Proof.* Apply Calderon-Zygmund decomposition (Lemma 1.6 in Chapter 3) to $f = \chi_A$. We obtain, by assumption (i), a sequence of dyadic cubes $\{Q^j\}$ such that

$$A \subset \cup_j Q^j \text{ except for a set of measure zero}$$

$$\delta \leq \frac{|A \cap Q^j|}{|Q^j|} < 2^n \delta$$

and

$$\frac{|A \cap \tilde{Q}^j|}{|\tilde{Q}^j|} < \delta,$$

for any predecessor $\tilde{Q}^j$ of $Q^j$. By assumption (ii) we have $\tilde{Q}^j \subset B$ for each $j$. Hence we obtain

$$A \subset \cup_j \tilde{Q}^j \subset B.$$

We relable $\{\tilde{Q}^j\}$ so that they are nonoverlapping. Therefore we get

$$|A| \leq \sum_i |A \cap \tilde{Q}^i| \leq \delta \sum_i |\tilde{Q}^i| \leq \delta |B|.$$

## §2. Harnack Inequality

The main result in this section is the following Harnack inequality.

**Theorem 2.1.** *Suppose $u$ belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in $B_1$ with $u \geq 0$ in $B_1$ for some $f \in C(B_1)$. Then there holds*

$$\sup_{B_{\frac{1}{2}}} u \leq C \left\{ \inf_{B_{\frac{1}{2}}} u + \|f\|_{L^n(B_1)} \right\}$$

*where $C$ is a positive constant depending only on $n, \lambda$ and $\Lambda$.*

The interior Hölder continuity of solutions is a direct consequence, whose proof is identical to that of Theorem 3.4 in Chapter 4.

**Corollary 2.2.** *Suppose $u$ belongs to $\mathcal{S}(\lambda,\Lambda,f)$ in $B_1$ for some $f\in C(B_1)$. Then $u\in C^\alpha(B_1)$ for some $\alpha\in(0,1)$ depending only on $n,\lambda$ and $\Lambda$. Moreover there holds*

$$|u(x)-u(y)|\le C|x-y|^\alpha\Big\{\sup_{B_1}|u|+\|f\|_{L^n(B_1)}\Big\}\quad \textit{for any } x,y\in B_{\frac12}$$

*where $C=C(n,\lambda,\Lambda)$ is a positive constant.*

For convenience we work in cubes instead of balls. We will prove the following result.

**Lemma 2.3.** *Suppose $u$ belongs to $\mathcal{S}(\lambda,\Lambda,f)$ in $Q_{4\sqrt{n}}$ with $u\ge 0$ in $Q_{4\sqrt{n}}$ for some $f\in C(Q_{4\sqrt{n}})$. Then there exist two positive constants $\varepsilon_0$ and $C$, depending only on $n,\lambda$ and $\Lambda$, such that if $\inf_{Q_{1/4}}u\le 1$ and $\|f\|_{L^n(Q_{4\sqrt{n}})}\le\varepsilon_0$ there holds $\sup_{Q_{\frac14}}u\le C$.*

Theorem 2.1 follows from Lemma 2.3 easily. For $u\in\mathcal{S}(\lambda,\Lambda,f)$ in $Q_{4\sqrt{n}}$ with $u\ge 0$ in $Q_{4\sqrt{n}}$, consider

$$u_\delta=\frac{u}{\inf_{Q_{1/4}}u+\delta+\frac{1}{\varepsilon_0}\|f\|_{L^n(Q_{4\sqrt{n}})}}$$

for $\delta>0$. We apply Lemma 2.3 to $u_\delta$ to get, after letting $\delta\to 0$,

$$\sup_{Q_{\frac14}}u\le C\Big\{\inf_{Q_{\frac14}}u+\|f\|_{L^n(Q_{4\sqrt{n}})}\Big\}.$$

Then Theorem 2.1 follows by a standard covering argument.

Now we begin to prove Lemma 2.3. The following result is the key ingredient. It claims that if solution is small somewhere in $Q_3$ then it is under control in a good portion of $Q_1$.

**Lemma 2.4.** *Suppose $u$ belongs to $\mathcal{S}^+(\lambda,\Lambda,f)$ in $B_{2\sqrt{n}}$ for some $f\in C(B_{2\sqrt{n}})$. Then there exist constants $\varepsilon_0>0$, $\mu\in(0,1)$ and $M>1$, depending only on $n,\lambda$ and $\Lambda$, such that if*

$$\begin{aligned} &u\ge 0 \ \textit{in}\ B_{2\sqrt{n}}\\ &\inf_{Q_3}u\le 1\\ &\|f\|_{L^n(B_{2\sqrt{n}})}\le\varepsilon_0\end{aligned}\tag{1}$$

*there holds*

$$|\{u\le M\}\cap Q_1|>\mu.$$

*Proof.* We will construct a function $g$, which is very concave outside $Q_1$, such that if we correct $u$ by $g$ the contact set occurs in $Q_1$. In other words we localize where contact occurs by choosing suitable functions.

Note $B_{1/4} \subset B_{1/2} \subset Q_1 \subset Q_3 \subset B_{2\sqrt{n}}$. Define $g$ in $B_{2\sqrt{n}}$ by

$$g(x) = -M\big(1 - \frac{|x|^2}{4n}\big)^{\beta}$$

for large $\beta > 0$ to be determined and some $M > 0$. We choose $M$, according to $\beta$, such that

$$g = 0 \text{ on } \partial B_{2\sqrt{n}} \quad \text{and} \quad g \le -2 \text{ in } Q_3. \tag{2}$$

Set $w = u + g$ in $B_{2\sqrt{n}}$. We will show by choosing $\beta$ large that

$$w \in \mathcal{S}^+(\lambda, \Lambda, f) \text{ in } B_{2\sqrt{n}} \setminus Q_1. \tag{3}$$

Suppose $\varphi$ is a quadratic polynomial with the property that $w - \varphi$ has a local minimum at $x_0 \in B_{2\sqrt{n}}$. Then $u - (\varphi - g)$ has a local minimum at $x_0 \in B_{2\sqrt{n}}$. By definitions of $\mathcal{S}^+(\lambda, \Lambda, f)$ and the Pucci's extremal operator $\mathcal{M}^-$ we have

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0) - D^2 g(x_0)) \le f(x_0)$$

or

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0)) + \mathcal{M}^-(\lambda, \Lambda, -D^2 g(x_0)) \le f(x_0)$$

where we used the property of $\mathcal{M}^-$. We will choose $\beta$ large such that

$$\mathcal{M}^-(\lambda, \Lambda, -D^2 g(x_0)) \ge 0 \text{ for any } x_0 \in B_{2\sqrt{n}} \setminus B_{\frac{1}{4}}.$$

We need to calculate the Hessian matrix of $g$. Note

$$D_{ij} g(x) = \frac{M}{2n}\beta\big(1 - \frac{|x|^2}{4n}\big)^{\beta-1}\delta_{ij} - \frac{M}{(2n)^2}\beta(\beta-1)\big(1 - \frac{|x|^2}{4n}\big)^{\beta-2} x_i x_j.$$

If we choose $x = (|x|, 0, \cdots, 0)$ then the eigenvalues of $-D^2 g(x)$ are given by

$$\frac{M}{2n}\beta\big(1 - \frac{|x|^2}{4n}\big)^{\beta-2}\big(\frac{2\beta-1}{4n}|x|^2 - 1\big)$$

with multiplicity 1 and

$$-\frac{M}{2n}\beta\big(1 - \frac{|x|^2}{4n}\big)^{\beta-1}$$

with multiplicity $n-1$. We choose $\beta$ large such that for $|x| \geq 1/4$ the first eigenvalue is positive and the rest negative, denoted by $e^+(x)$ and $e^-(x)$ respectively. Therefore for $|x| \geq 1/4$ we have

$$\begin{aligned}
&\mathcal{M}^-(\lambda, \Lambda, -D^2 g(x)) = \lambda e^+(x) + (n-1)\Lambda e^-(x) \\
=&\frac{M}{2n}\beta\big(1 - \frac{|x|^2}{4n}\big)^{\beta-2}\Big\{\lambda\big(\frac{2\beta-1}{4n}|x|^2 - 1\big) - (n-1)\Lambda\big(1 - \frac{|x|^2}{4n}\big)\Big\} \geq 0
\end{aligned}$$

if we choose $\beta$ large, depending only on $n, \lambda$ and $\Lambda$. This finishes the proof of (3). In fact we obtain

$$w \in \mathcal{S}^+(\lambda, \Lambda, f + \eta) \text{ in } B_{2\sqrt{n}}$$

for some $\eta \in C_0^\infty(Q_1)$ and $0 \leq \eta \leq C(n, \lambda, \Lambda)$.

We may apply Theorem 1.2 to $w$ in $B_{2\sqrt{n}}$. Note $\inf_{Q_3} w \leq -1$ and $w \geq 0$ on $\partial B_{2\sqrt{n}}$ by (1) and (2). We obtain

$$\begin{aligned}
1 \leq C&\Big(\int_{B_{2\sqrt{n}} \cap \{w = \Gamma_w\}} (|f| + \eta)^n\Big)^{\frac{1}{n}} \\
&\leq C\|f\|_{L^n(B_{2\sqrt{n}})} + C|\{w = \Gamma_w\} \cap Q_1|^{\frac{1}{n}}.
\end{aligned}$$

Choosing $\varepsilon_0$ small enough we get

$$\frac{1}{2} \leq C|\{w = \Gamma_w\} \cap Q_1|^{\frac{1}{n}} \leq C|\{u \leq M\} \cap Q_1|^{\frac{1}{n}}$$

since $w(x) = \Gamma_w(x)$ implies $w(x) \leq 0$ and hence $u(x) \leq -g(x) \leq M$. This finishes the proof.

Next we prove the power decay of distribution functions.

**Lemma 2.5.** *Suppose $u$ belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for some $f \in C(B_{2\sqrt{n}})$. Then there exist positive constants $\varepsilon_0$, $\varepsilon$ and $C$, depending only on $n, \lambda$ and $\Lambda$, such that if*

$$\begin{aligned}
&u \geq 0 \text{ in } B_{2\sqrt{n}} \\
&\inf_{Q_3} u \leq 1 \qquad\qquad (1)\\
&\|f\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0
\end{aligned}$$

*there holds*

$$|\{u \geq t\} \cap Q_1| \leq Ct^{-\varepsilon} \text{ for } t > 0.$$

*Proof.* We will prove that under the assumption (1) there holds

$$|\{u > M^k\} \cap Q_1| \leq (1-\mu)^k \text{ for } k = 1, 2, \cdots, \tag{2}$$

where $M$ and $\mu$ are as in Lemma 2.4.

For $k = 1$, (2) is just Lemma 2.4. Suppose now (2) holds for $k-1$. Set

$$A = \{u > M^k\} \cap Q_1, \qquad B = \{u > M^{k-1}\} \cap Q_1.$$

We will use Lemma 1.3 to prove that

$$|A| \leq (1-\mu)|B|. \tag{3}$$

Clearly $A \subset B \subset Q_1$ and $|A| \leq |\{u > M\} \cap Q_1| \leq 1 - \mu$ by Lemma 2.4. We claim that if $Q = Q_r(x_0)$ is a cube in $Q_1$ such that

$$|A \cap Q| > (1-\mu)|Q| \tag{4}$$

then $\tilde{Q} \cap Q_1 \subset B$ for $\tilde{Q} = Q_{3r}(x_0)$. We prove it by contradiction. Suppose not. We may take $\tilde{x} \in \tilde{Q}$ such that $u(\tilde{x}) \leq M^{k-1}$. Consider the transformation

$$x = x_0 + ry \quad \text{for } y \in Q_1 \text{ and } x \in Q = Q_r(x_0)$$

and the function

$$\tilde{u}(y) = \frac{1}{M^{k-1}} u(x).$$

Then $\tilde{u} \geq 0$ in $B_{2\sqrt{n}}$ and $\inf_{Q_3} \tilde{u} \leq 1$. It is easy to check that $\tilde{u} \in \mathcal{S}^+(\lambda, \Lambda, \tilde{f})$ in $B_{2\sqrt{n}}$ with $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0$. In fact we have

$$\tilde{f}(y) = \frac{r^2}{M^{k-1}} f(x) \text{ for } y \in B_{2\sqrt{n}}$$

and hence

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \frac{r}{M^{k-1}} \|f\|_{L^n(B_{2\sqrt{n}})} \leq \|f\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0.$$

Hence $\tilde{u}$ satisfies the assumption (1). We may apply Lemma 2.4 to $\tilde{u}$ to get

$$\mu < |\{\tilde{u}(y) \leq M\} \cap Q_1| = r^{-n} |\{u(x) \leq M^k\} \cap Q|.$$

Hence $|Q \cap A^C| > \mu|Q|$, which contradicts (4). We are in a position to apply Lemma 1.3 to get (3).

*Proof of Lemma 2.3.* We prove that there exist two constants $\theta > 1$ and $M_0 >> 1$, depending only on $n, \lambda$ and $\Lambda$, such that if $u(x_0) = P > M_0$ for some $x_0 \in B_{1/4}$ there exists a sequence $\{x_k\} \in B_{1/2}$ such that

$$u(x_k) \geq \theta^k P \text{ for } k = 0, 1, 2, \cdots .$$

This contradicts the boundedness of $u$ and hence we conclude that $\sup_{B_{\frac{1}{4}}} u \leq M_0$.

Suppose $u(x_0) = P > M_0$ for some $x_0 \in B_{1/4}$. We will determine $M_0$ and $\theta$ in the process. Consider a cube $Q_r(x_0)$, centered at $x_0$ with side length $r$, which will be chosen later. We want to find a point $x_1 \in Q_{4\sqrt{n}r}(x_0)$ such that $u(x_1) \geq \theta P$. To do that we first choose $r$ such that $\{u > P/2\}$ covers less than half of $Q_r(x_0)$. This can be done by using the power decay of the distribution function of $u$.

Note $\inf_{Q_3} u \leq \inf_{Q_{1/4}} u \leq 1$. Hence Lemma 2.5 implies

$$|\{u > \frac{P}{2}\} \cap Q_1| \leq C(\frac{P}{2})^{-\varepsilon}.$$

We choose $r$ such that $r^n/2 \geq C(P/2)^{-\varepsilon}$ and $r \leq 1/4$. Hence we have, for such $r$, $Q_r(x_0) \subset Q_1$ and

$$\frac{1}{|Q_r(x_0)|}|\{u > P/2\} \cap Q_r(x_0)| \leq \frac{1}{2}. \tag{1}$$

Next we show that for $\theta > 1$, with $\theta - 1$ small, $u \geq \theta P$ at some point in $Q_{4\sqrt{n}r}(x_0)$. We prove it by contradiction. Suppose $u \leq \theta P$ in $Q_{4\sqrt{n}r}(x_0)$. Consider the transformation

$$x = x_0 + ry \quad \text{for } y \in Q_{4\sqrt{n}} \text{ and } x \in Q_{4\sqrt{n}r}(x_0)$$

and the function

$$\tilde{u}(y) = \frac{\theta P - u(x)}{(\theta - 1)P}.$$

Obviously $\tilde{u} \geq 0$ in $B_{2\sqrt{n}}$ and $\tilde{u}(0) = 1$, hence $\inf_{Q_3} \tilde{u} \leq 1$. It is easy to check that $\tilde{u} \in \mathcal{S}^+(\lambda, \Lambda, \tilde{f})$ in $B_{2\sqrt{n}}$ with $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0$. In fact we have

$$\tilde{f}(y) = -\frac{r^2}{(\theta - 1)P} f(x) \text{ for } y \in B_{2\sqrt{n}}$$

and hence

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \frac{r}{(\theta - 1)P}\|f\|_{L^n(B_{2\sqrt{n}r}(x_0))} \leq \varepsilon_0$$

if we choose $P$ such that $r \le (\theta-1)P$. Hence we may apply Lemma 2.4 to $\tilde{u}$. Note that $u(x) \le P/2$ if and only if $\tilde{u}(y) \ge \frac{\theta-1/2}{\theta-1}$ and that $\frac{\theta-1/2}{\theta-1}$ is large if $\theta$ is close to 1. So we obtain

$$\frac{1}{|Q_r(x_0)|}|\{u \le P/2\} \cap Q_r(x_0)| = |\{\tilde{u} \ge \frac{\theta-1/2}{\theta-1}\} \cap Q_1| \le C\Big(\frac{\theta-1/2}{\theta-1}\Big)^{-\varepsilon} < \frac{1}{2}$$

if $\theta$ is chosen close to 1. This contradicts (1).

Hence we conclude that there exists a $\theta = \theta(n, \lambda, \Lambda) > 1$ such that if

$$u(x_0) = P \text{ for some } \; x_0 \in B_{\frac{1}{4}}$$

then

$$u(x_1) \ge \theta P \text{ for some } \; x_1 \in Q_{4\sqrt{n}r}(x_0) \subset B_{2nr}(x_0)$$

provided

$$C(n, \lambda, \Lambda)P^{-\frac{\varepsilon}{n}} \le r \le (\theta-1)P.$$

So we need to choose $P$ such that $P \ge \big(\frac{C}{\theta-1}\big)^{\frac{n}{n+\varepsilon}}$ and then take $r = CP^{-\frac{\varepsilon}{n}}$.

Now we may iterate the above result to get a sequence $\{x_k\}$ such that for any $k = 1, 2, \cdots$,

$$u(x_k) \ge \theta^k P \text{ for some } \; x_k \in B_{2nr_k}(x_{k-1})$$

where $r_k = C(\theta^{k-1}P)^{-\frac{\varepsilon}{n}} = C\theta^{-(k-1)\frac{\varepsilon}{n}}P^{-\frac{\varepsilon}{n}}$. In order to have $\{x_k\} \in B_{1/2}$ we need $\sum 2nr_k < 1/4$. Hence we choose $M_0$ such that

$$M_0^{\frac{\varepsilon}{n}} \ge 8nC\sum_{k=1}^{\infty}\theta^{-(k-1)\frac{\varepsilon}{n}} \quad \text{and} \quad M_0 \ge \big(\frac{C}{\theta-1}\big)^{\frac{n}{n+\varepsilon}}$$

and then take $P > M_0$. This finishes the proof.

In the rest of this section we prove a technical lemma concerning the second order derivatives of functions in $\mathcal{S}(\lambda, \Lambda, f)$. Such result will be needed in the discussion of $W^{2,p}$ estimates. First we introduce some terminology.

Let $\Omega$ be a bounded domain and $u$ be a continuous function in $\Omega$. We define for $M > 0$

$$G_M^-(u, \Omega) = \{x_0 \in \Omega; \text{ there exists an affine function } L \text{ such that}$$
$$L(x) - \frac{M}{2}|x-x_0|^2 \le u(x) \text{ for } x \in \Omega \text{ with equality at } x_0\}$$
$$G_M^+(u, \Omega) = \{x_0 \in \Omega; \text{ there exists an affine function } L \text{ such that}$$
$$L(x) + \frac{M}{2}|x-x_0|^2 \ge u(x) \text{ for } x \in \Omega \text{ with equality at } x_0\}$$
$$G_M(u, \Omega) = G_M^+(u, \Omega) \cap G_M^-(u, \Omega).$$

We also define

$$
\begin{aligned}
A_M^-(u,\Omega) =& \Omega \setminus G_M^-(u,\Omega)\\
A_M^+(u,\Omega) =& \Omega \setminus G_M^+(u,\Omega)\\
A_M(u,\Omega) =& \Omega \setminus G_M(u,\Omega).
\end{aligned}
$$

In other words $G_M^-(u,\Omega)$ (resp. $G_M^+(u,\Omega)$) consists of points where there is a concave (resp. convex) paraboloid of opening $M$ touching $u$ from below (resp. above). Intuitively $|A_M(u,\Omega)|$ behaves like the distribution function of $D^2u$. Hence for integrability of $D^2u$ we need to study the decay of $|A_M(u,\Omega)|$.

**Lemma 2.6.** *Suppose that $\Omega$ is a bounded domain with $B_{6\sqrt{n}} \subset \Omega$ and that $u$ belongs to $\mathcal{S}^+(\lambda,\Lambda,f)$ in $B_{6\sqrt{n}}$ for some $f \in C(B_{6\sqrt{n}})$. Then there exist positive constants $\delta_0$, $\mu$ and $C$, depending only on $n, \lambda$ and $\Lambda$, such that if $|u| \le 1$ in $\Omega$ and $\|f\|_{L^n(B_{6\sqrt{n}})} \le \delta_0$ there holds*

$$|A_t^-(u,\Omega) \cap Q_1| \le Ct^{-\mu} \text{ for any } t > 0.$$

*If, in addition, $u \in \mathcal{S}(\lambda,\Lambda,f)$ in $B_{6\sqrt{n}}$, then*

$$|A_t(u,\Omega) \cap Q_1| \le Ct^{-\mu} \text{ for any } t > 0.$$

In the proof of Lemma 2.6 we need the *maximal functions* of local integrable functions. For $g \in L^1_{loc}(\mathbb{R}^n)$ we define

$$m(g)(x) = \sup_{r>0} \frac{1}{|Q_r(x)|}\int_{Q_r(x)} |g|.$$

The maximal operator $m$ is of weak type (1,1) and of strong type $(p,p)$ for $1 < p \le \infty$, i.e.,

$$
\begin{aligned}
&|\{x \in \mathbb{R}^n; m(g)(x) \ge t\}| \le \frac{c_1(n)}{t}\|g\|_{L^1(\mathbb{R}^n)} \quad \text{for any } t > 0\\
&\|m(g)\|_{L^p(\mathbb{R}^n)} \le c_2(n,p)\|g\|_{L^p(\mathbb{R}^n)} \quad \text{for } \ 1 < p \le \infty.
\end{aligned}
$$

Now we begin to prove Lemma 2.6. The following result is the key ingredient. It claims that if $u$ has a tangent paraboloid with opening 1 from below somewhere in $Q_3$ then the set where $u$ has a tangent paraboloid from below with opening $M$ in $Q_1$ is large. Compare it with Lemma 2.4.

**Lemma 2.7.** *Suppose that $\Omega$ is a bounded domain with $B_{6\sqrt{n}} \subset \Omega$ and that $u$ belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{6\sqrt{n}}$ for some $f \in C(B_{6\sqrt{n}})$. Then there exist constants $0 < \sigma < 1$, $\delta_0 > 0$ and $M > 1$, depending only on $n, \lambda$ and $\Lambda$, such that if $\|f\|_{L^n(B_{6\sqrt{n}})} \leq \delta_0$ and $G_1^-(u, \Omega) \cap Q_3 \neq \phi$, then*

$$|G_M^-(u, \Omega) \cap Q_1| \geq 1 - \sigma.$$

*Proof.* Since $G_1^-(u, \Omega) \cap Q_3 \neq \phi$, there is an affine function $L_1$ such that

$$v \geq P_1 \text{ in } \Omega \text{ with equality at some point in } Q_3$$

where

$$v(x) = \frac{u(x)}{2n} + L_1(x) \quad \text{and} \quad P_1(x) = 1 - \frac{|x|^2}{4n}.$$

This implies $v \geq 0$ in $B_{2\sqrt{n}}$ and $\inf_{Q_3} v \leq 1$. Then as in the proof of Lemma 2.4, for $w = v + g$, where $g$ in the function constructed in Lemma 2.4, we have

$$|\{w = \Gamma_w\} \cap Q_1| \geq 1 - \sigma$$

for some $\sigma \in (0, 1)$ if $\delta_0$ is chosen small. Now we need to prove $\{w = \Gamma_w\} \cap Q_1 \subset G_M^-(u, \Omega) \cap Q_1$ for some $M > 1$. Let $x_0 \in \{w = \Gamma_w\} \cap Q_1$ and take an affine function $L_2$ with $L_2 < 0$ on $\partial B_{2\sqrt{n}}$ and

$$L_2 \leq \Gamma_w \leq v + g \text{ in } B_{2\sqrt{n}} \text{ with equality at } x_0.$$

It follows that

$$(1) \qquad P_2 \leq L_2 - g \leq v \text{ in } B_{2\sqrt{n}} \text{ with equality at } x_0$$

for a concave paraboloid $P_2$ of opening $M_0 = M_0(n, \lambda, \Lambda) > 0$.

Next we prove $P_2 \leq v$ in $\Omega \setminus B_{2\sqrt{n}}$. Note that $P_2 < -g = 0 = P_1$ on $\partial B_{2\sqrt{n}}$ and that $P_2(x_0) = v(x_0) \geq P_1(x_0)$ with $x_0 \in Q_1 \subset B_{2\sqrt{n}}$. If we take $M_0 > 1/(2n)$, then $\{P_2 - P_1 \geq 0\}$ is convex. We conclude that $P_2 - P_1 < 0$ in $\mathbb{R}^n \setminus B_{2\sqrt{n}}$. Hence we have $P_2 \leq P_1 \leq v$ in $\Omega \setminus B_{2\sqrt{n}}$. By (1) and the definition of $v$, we get $x_0 \in G_{2nM_0}^-(u, \Omega) \cap Q_1$ with $2nM_0 > 1$.

*Proof of Lemma 2.6.* Recall $B_{6\sqrt{n}} \subset \Omega$, $u \in \mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{6\sqrt{n}}$ and

$$(1) \qquad |u|_{L^\infty(\Omega)} \leq 1, \qquad \|f\|_{L^n(B_{6\sqrt{n}})} \leq \delta_0.$$

We will prove that there exist constants $M > 1$ and $0 < \gamma < 1$, depending only on $n, \lambda$ and $\Lambda$, such that

$$|A_{M^k}^-(u, \Omega) \cap Q_1| \leq \gamma^k \text{ for any } k = 0, 1, \cdots.$$

Step I. There exist constants $M > 1$ and $0 < \sigma < 1$ such that

$$|G^-_M(u, \Omega) \cap Q_1| \geq 1 - \sigma. \tag{2}$$

It is easy to see that $|u|_{L^\infty(\Omega)} \leq 1$ implies that

$$G^-_{c(n)}(u, \Omega) \cap Q_3 \neq \phi$$

for some constant $c(n)$ depending only on $n$. We may apply Lemma 2.6 to $u/c(n)$ to get (2). By a simple adjustment we may assume that $\delta_0, M$ and $\sigma$ in Step I are the same as those in Lemma 2.6.

Step II. We extend $f$ by zero outside $B_{6\sqrt{n}}$ and set for $k = 0, 1, \cdots$,

$$\begin{aligned} A &= A^-_{M^{k+1}}(u, \Omega) \cap Q_1 \\ B &= \left(A^-_{M^k}(u, \Omega) \cap Q_1\right) \cup \left\{x \in Q_1; m(f^n)(x) \geq (c_1 M^k)^n\right\} \end{aligned}$$

for some $c_1 > 0$ to be determined. Then there holds

$$|A| \leq \sigma |B|$$

where $M > 1$ and $0 < \sigma < 1$ are as before. Recall that $m(f^n)$ denotes the maximal function of $f^n$.

We prove it by Lemma 1.3. It is easy to see $|A| \leq \sigma$ since we have $|G^-_{M^{k+1}}(u, \Omega) \cap Q_1| \geq |G^-_M(u, \Omega) \cap Q_1| \geq 1 - \sigma$ by Step I. Next we claim that if $Q = Q_r(x_0)$ is a cube in $Q_1$ such that

$$|A^-_{M^{k+1}}(u, \Omega) \cap Q| = |A \cap Q| > \sigma |Q| \tag{3}$$

then $\tilde{Q} \cap Q_1 \subset B$ for $\tilde{Q} = Q_{3r}(x_0)$. We prove it by contradiction. Suppose not. We may take an $\tilde{x}$ such that

$$\tilde{x} \in G^-_{M^k}(u, \Omega) \cap \tilde{Q}$$

and

$$\sup_{r > 0} \frac{1}{|Q_r(\tilde{x})|} \int_{Q_r(\tilde{x})} |f|^n \leq (c_1 M^k)^n.$$

Consider the transformation

$$x = x_0 + ry \quad \text{for } y \in Q_1 \text{ and } x \in Q = Q_r(x_0)$$

and the function

$$\tilde{u}(y) = \frac{1}{r^2 M^k} u(x).$$

It is easy to check that $B_{6\sqrt{n}} \subset \tilde{\Omega}$, the image of $\Omega$ under the transformation above, and that $\tilde{u} \in \mathcal{S}^+(\lambda, \Lambda, \tilde{f})$ in $B_{6\sqrt{n}}$ with

$$\tilde{f}(y) = \frac{1}{M^k} f(x) \text{ for } y \in B_{6\sqrt{n}}.$$

By the choice of $\tilde{x}$ we have

$$G_1^-(\tilde{u}, \tilde{\Omega}) \cap Q_3 \neq \phi.$$

Since $B_{6\sqrt{n}r}(x_0) \subset Q_{15\sqrt{n}r}(\tilde{x})$ there holds

$$\|\tilde{f}\|_{L^n(B_{6\sqrt{n}})} \leq \frac{1}{rM^k} \|f\|_{L^n(Q_{15\sqrt{n}r}(\tilde{x}))} \leq c(n)c_1 \leq \delta_0$$

if we take $c_1$ small enough, depending only on $n, \lambda$ and $\Lambda$.

Hence $\tilde{u}$ satisfies the assumption of Lemma 2.7 with $\Omega$ replaced by $\tilde{\Omega}$. We may apply Lemma 2.7 to $\tilde{u}$ to get

$$|G_M^-(\tilde{u}, \tilde{\Omega}) \cap Q_1| \geq 1 - \sigma$$

or

$$|G_{M^{k+1}}^-(u, \Omega) \cap Q| > (1 - \sigma)|Q|.$$

This contradicts (3). We are in a position to apply Lemma 1.3.

Step III. We finish the proof of Lemma 2.6. Define for $k = 0, 1, \cdots$,

$$\alpha_k = |A_{M^k}^-(u, \Omega) \cap Q_1|$$
$$\beta_k = |\{x \in Q_1; m(f^n)(x) \geq (c_1 M^k)^n\}|.$$

Then Step II implies $\alpha_{k+1} \leq \sigma(\alpha_k + \beta_k)$ for any $k = 0, 1, \cdots$. Hence by iteration we have

$$\alpha_k \leq \sigma^k + \sum_{i=0}^{k-1} \sigma^{k-i} \beta_i.$$

Since $\|f^n\|_{L^1} \leq \delta_0^n$ and the maximal operator is of weak type (1, 1), we conclude that

$$\beta_k \leq c(n)\delta_0^n (c_1 M^k)^{-n} = C(n, \lambda, \Lambda) M^{-nk}.$$

This implies

$$\sum_{i=0}^{k-1} \sigma^{k-i} \beta_i \leq C \sum_{i=0}^{k-1} \sigma^{k-i} M^{-ni} \leq Ck\gamma_0^k$$

with $\gamma_0 = \max\{\sigma, M^{-n}\} < 1$. Therefore we obtain for $k$ large

$$\alpha_k \le \sigma^k + Ck\gamma_0^k \le (1 + Ck)\gamma_0^k \le \gamma^k$$

for some $\gamma = \gamma(n, \lambda, \Lambda) \in (0, 1)$. This finishes the proof.

*Remark.* The polynomial decay of the function

$$\mu(t) = |A_t(u, \Omega) \cap Q_1|$$

for $u \in \mathcal{S}(\lambda, \Lambda, f)$ implies that $D^2u$ is $L^p$-integrable in $Q_1$ for small $p > 0$, depending only on $n, \lambda$ and $\Lambda$. In order to show the $L^p$-integrability for large $p$ we need to speed up the convergence in the proof of Lemma 2.6. We will discuss $W^{2,p}$ estimates in Section 4.

## §3. Schauder Estimates

In this section we will prove the Schauder estimates for viscosity solutions.

Throughout this section we always assume that $a_{ij} \in C(B_1)$ satisfies

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \text{ for any } x \in B_1 \text{ and any } \xi \in \mathbb{R}^n$$

for some positive constants $\lambda$ and $\Lambda$ and that $f$ is a continuous function in $B_1$.

The following approximation result plays an important role in the discussion of regularity theory.

**Lemma 3.1.** *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_1$$

*with $|u| \le 1$ in $B_1$. Assume for some $0 < \varepsilon < 1/16$,*

$$\|a_{ij} - a_{ij}(0)\|_{L^n(B_{3/4})} \le \varepsilon.$$

*Then there exists a function $h \in C(\bar{B}_{3/4})$ with $a_{ij}(0)D_{ij}h = 0$ in $B_{3/4}$ and $|h| \le 1$ in $B_{3/4}$ such that*

$$|u - h|_{L^\infty(B_{\frac{1}{2}})} \le C\Big\{\varepsilon^\gamma + \|f\|_{L^n(B_1)}\Big\}$$

*where $C = C(n, \lambda, \Lambda)$ is a positive constant and $\gamma = \gamma(n, \lambda, \Lambda) \in (0, 1)$.*

*Proof.* Solve for $h \in C(\bar{B}_{3/4}) \cap C^\infty(B_{3/4})$ such that

$$\begin{aligned} a_{ij}(0)D_{ij}h &= 0 \quad \text{in } B_{3/4} \\ h &= u \quad \text{on } \partial B_{3/4}. \end{aligned}$$

The maximum principle implies $|h| \le 1$ in $B_{3/4}$. Note that $u$ belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in $B_1$. Corollary 2.2 implies that $u \in C^\alpha(\bar{B}_{3/4})$ for some $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$ with the estimate

$$\|u\|_{C^\alpha(\bar{B}_{3/4})} \le C(n, \lambda, \Lambda)\big\{1 + \|f\|_{L^n(B_1)}\big\}.$$

By Lemma 3.6 in Chapter 1 we have

$$\|h\|_{C^{\frac{\alpha}{2}}(\bar{B}_{3/4})} \le C\|u\|_{C^\alpha(\bar{B}_{3/4})} \le C(n, \lambda, \Lambda)\big\{1 + \|f\|_{L^n(B_1)}\big\}.$$

Since $u - h = 0$ on $\partial B_{3/4}$ we get for any $0 < \delta < 1/4$

$$|u - h|_{L^\infty(\partial B_{\frac{3}{4}-\delta})} \le C\delta^{\frac{\alpha}{2}}\big\{1 + \|f\|_{L^n(B_1)}\big\}. \tag{1}$$

We claim for any $0 < \delta < 1$

$$|D^2 h|_{L^\infty(B_{\frac{3}{4}-\delta})} \le C\delta^{\frac{\alpha}{2}-2}\big\{1 + \|f\|_{L^n(B_1)}\big\}. \tag{2}$$

In fact for any $x_0 \in B_{3/4-\delta}$, we apply interior $C^2$-estimate to $h - h(x_1)$ in $B_\delta(x_0) \subset B_{3/4}$ for some $x_1 \in \partial B_\delta(x_0)$ and obtain

$$|D^2 h(x_0)| \le C\delta^{-2} \sup_{B_\delta(x_0)} |h - h(x_1)| \le C\delta^{-2}\delta^{\frac{\alpha}{2}}\big\{1 + \|f\|_{L^n(B_1)}\big\}.$$

Note that $u - h$ is a viscosity solution of

$$a_{ij}D_{ij}(u - h) = f - (a_{ij} - a_{ij}(0))D_{ij}h \equiv F \quad \text{in } B_{3/4}.$$

By Theorem 1.2 (Alexandroff maximum principle) we have with (1) and (2)

$$\begin{aligned} &|u - h|_{L^\infty(B_{\frac{3}{4}-\delta})} \le |u - h|_{L^\infty(\partial B_{\frac{3}{4}-\delta})} + C\|F\|_{L^n(B_{\frac{3}{4}-\delta})} \\ \le&|u - h|_{L^\infty(\partial B_{\frac{3}{4}-\delta})} + C|D^2 h|_{L^\infty(B_{\frac{3}{4}-\delta})}\|a_{ij} - a_{ij}(0)\|_{L^n(B_{\frac{3}{4}})} + C\|f\|_{L^n(B_1)} \\ \le&C(\delta^{\frac{\alpha}{2}} + \delta^{\frac{\alpha}{2}-2}\varepsilon)\big\{1 + \|f\|_{L^n(B_1)}\big\} + C\|f\|_{L^n(B_1)}. \end{aligned}$$

Take $\delta = \varepsilon^{1/2} < 1/4$ and then $\gamma = \alpha/4$. This finishes the proof.

For the next result we need to introduce the following concept.

**Definition.** A function $g$ is Hölder continuous at 0 with exponent $\alpha$ in $L^n$-sence if

$$[g]_{C^\alpha_{L^n}}(0) \equiv \sup_{0<r<1} \frac{1}{r^\alpha}\left(\frac{1}{|B_r|}\int_{B_r} |g - g(0)|^n\right)^{\frac{1}{n}} < \infty.$$

Now we state the Schauder estimates.

**Theorem 3.2.** *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_1.$$

*Assume $\{a_{ij}\}$ is Hölder continuous at 0 with exponent $\alpha$ in $L^n$-sence for some $\alpha \in (0,1)$. If $f$ is Hölder continuous at 0 with exponent $\alpha$ in $L^n$-sence, then $u$ is $C^{2,\alpha}$ at 0. Moreover there exists a polynomial $P$ of degree 2 such that*

$$|u - P|_{L^\infty(B_r(0))} \le C_* r^{2+\alpha} \text{ for any } 0 < r < 1$$
$$|P(0)| + |DP(0)| + |D^2P(0)| \le C_*$$

*and*

$$C_* \le C\left\{ |u|_{L^\infty(B_1)} + |f(0)| + [f]_{C^\alpha_{L^n}}(0) \right\}$$

*where $C$ is a positive constant depending only on $n, \lambda, \Lambda, \alpha$ and $[a_{ij}]_{C^\alpha_{L^n}}(0)$.*

*Proof.* First we assume $f(0) = 0$. For that we may consider $v = u - b_{ij}x_ix_jf(0)/2$ for some constant matrix $\{b_{ij}\}$ such that $a_{ij}(0)b_{ij} = 1$. By scaling we also assume that $[a_{ij}]_{C^\alpha_{L^n}}(0)$ is small. Next by considering for $\delta > 0$

$$\frac{u}{|u|_{L^\infty(B_1)} + \frac{1}{\delta}[f]_{C^\alpha_{L^n}}(0)}$$

we may assume $|u|_{L^\infty(B_1)} \le 1$ and $[f]_{C^\alpha_{L^n}}(0) \le \delta$.

In the following we prove that there is a constant $\delta > 0$, depending only on $n, \lambda, \Lambda$ and $\alpha$, such that if $u \in C(B_1)$ is a viscosity solution of

$$a_{ij}D_{ij}u = f \quad \text{in } B_1$$

with

$$|u|_{L^\infty(B_1)} \le 1, \qquad [a_{ij}]_{C^\alpha_{L^n}}(0) \le \delta$$
$$\left( \frac{1}{|B_r|} \int_{B_r} |f|^n \right)^{\frac{1}{n}} \le \delta r^\alpha \text{ for any } 0 < r < 1$$

then there exists a polynomial $P$ of degree 2 such that

$$|u - P|_{L^\infty(B_r(0))} \le C r^{2+\alpha} \text{ for any } 0 < r < 1 \tag{1}$$

and

$$|P(0)| + |DP(0)| + |D^2P(0)| \le C \tag{2}$$

for some positive constant $C$ depending only on $n, \lambda, \Lambda$ and $\alpha$.

We claim that there exist $0 < \mu < 1$, depending only on $n, \lambda, \Lambda$ and $\alpha$, and a sequence of polynomials of degree 2

$$P_k(x) = a_k + b_k \cdot x + \frac{1}{2} x^t C_k x$$

such that for any $k = 0, 1, 2, \cdots$,

$$(3) \qquad \begin{aligned} &a_{ij}(0) D_{ij} P_k = 0 \\ &|u - P_k|_{L^\infty(B_{\mu^k})} \le \mu^{k(2+\alpha)} \end{aligned}$$

and

$$(4) \qquad |a_k - a_{k-1}| + \mu^{k-1}|b_k - b_{k-1}| + \mu^{2(k-1)}|C_k - C_{k-1}| \le C\mu^{(k-1)(2+\alpha)}$$

where $P_0 = P_{-1} \equiv 0$ and $C$ is a positive constant, depending only on $n, \lambda, \Lambda$ and $\alpha$.

We first prove that Theorem 3.2 follows from (3) and (4). It is easy to see that $a_k, b_k$ and $C_k$ converge and the limiting polynomial

$$p(x) = a_\infty + b_\infty \cdot x + \frac{1}{2} x^t C_\infty x$$

satisfies

$$|P_k(x) - p(x)| \le C\big\{|x|^2 \mu^{\alpha k} + |x| \mu^{(\alpha+1)k} + \mu^{(\alpha+2)k}\big\} \le C\mu^{(2+\alpha)k}$$

for any $|x| \le \mu^k$. Hence we have for $|x| \le \mu^k$

$$|u(x) - p(x)| \le |u(x) - P_k(x)| + |P_k(x) - p(x)| \le C\mu^{(2+\alpha)k}$$

which implies that

$$|u(x) - p(x)| \le C|x|^{2+\alpha} \quad \text{for any } x \in B_1.$$

Now we prove (3) and (4). Clearly (3) and (4) hold for $k = 0$. Assume they hold for $k = 0, 1, 2, \cdots, l$. We prove for $k = l + 1$. Consider the function

$$\tilde{u}(y) = \frac{1}{\mu^{l(2+\alpha)}} (u - P_l)(\mu^l y) \text{ for } y \in B_1.$$

Then $\tilde{u} \in C(B_1)$ is a viscosity solution of

$$\tilde{a}_{ij} D_{ij} \tilde{u} = \tilde{f} \quad \text{in } B_1$$

with

$$\tilde{a}_{ij}(y) = \frac{1}{\mu^{l\alpha}} a_{ij}(\mu^l y)$$
$$\tilde{f}(y) = \frac{1}{\mu^{l\alpha}} \big\{ f(\mu^l y) - a_{ij}(\mu^l y) D_{ij} P_k \big\}.$$

Now we check that $\tilde{u}$ satisfies the assumptions of Lemma 3.1. For that we calculate

$$\|\tilde{a}_{ij} - \tilde{a}_{ij}(0)\|_{L^n(B_1)} \le \frac{1}{\mu^{l\alpha}} \|a_{ij} - a_{ij}(0)\|_{L^n(B_{\mu^l})} \le [a_{ij}]_{C^\alpha_{L^n}}(0) \le \delta$$

and

$$\|\tilde{f}\|_{L^n(B_1)} \le \frac{1}{\mu^{l\alpha}} \|f\|_{L^n(B_{\mu^l})} + \frac{1}{\mu^{l\alpha}} \sup |D^2 P_l| \|a_{ij} - a_{ij}(0)\|_{L^n(B_{\mu^l})} \le \delta + C\delta$$

where we used

$$|D^2 P_l| \le \sum_{k=1}^{l} |D^2 P_k - D^2 P_{k-1}| \le \sum_{k=1}^{l} \mu^{(k-1)\alpha} \le C.$$

Hence we take $\varepsilon = C(n, \lambda, \Lambda)\delta$ in Lemma 3.1. Then by Lemma 3.1 there exists a function $h \in C(\bar{B}_{3/4})$ with $\tilde{a}_{ij}(0) D_{ij} h = 0$ in $B_{3/4}$ and $|h| \le 1$ in $B_{3/4}$ such that

$$|\tilde{u} - h|_{L^\infty(B_{\frac{1}{2}})} \le C\Big\{ \varepsilon^\gamma + \varepsilon \Big\} \le 2C\varepsilon^\gamma.$$

Write $\tilde{P}(y) = h(0) + Dh(0) + y^t D^2 h(0) y/2$. Then by interior estimates for $h$ we have

$$|\tilde{u} - \tilde{P}|_{L^\infty(B_\mu)} \le |\tilde{u} - h|_{L^\infty(B_\mu)} + |h - \tilde{P}|_{L^\infty(B_\mu)} \le 2C\varepsilon^\gamma + C\mu^3 \le \mu^{2+\alpha}$$

by choosing $\mu$ small and then $\varepsilon$ small accordingly. Rescaling back we have

$$|u(x) - P_l(x) - \mu^{l(2+\alpha)} \tilde{P}(\mu^{-l} x)| \le \mu^{(l+1)(2+\alpha)} \text{ for any } x \in B_{\mu^{l+1}}.$$

This implies (3) for $k = l + 1$, if we define

$$P_{k+1}(x) = P_k(x) + \mu^{l(2+\alpha)} \tilde{P}(\mu^{-l} x).$$

The estimate (4) follows easily.

To finish this section we state the Cordes-Nirenberg type estimate. The proof is similar to that of Theorem 3.2.

**Theorem 3.3.** *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \textit{in } B_1.$$

*Then for any $\alpha \in (0,1)$ there exists an $\theta > 0$, depending only on $n, \lambda, \Lambda$ and $\alpha$, such that if*

$$\left(\frac{1}{|B_r|}\int_{B_r}|a_{ij} - a_{ij}(0)|^n\right)^{\frac{1}{n}} \le \theta \quad \textit{for any } 0 < r \le 1,$$

*then $u$ is $C^{1,\alpha}$ at 0; that is, there exists an affine function $L$ such that*

$$|u - L|_{L^\infty(B_r(0))} \le C_* r^{1+\alpha} \textit{ for any } 0 < r < 1$$
$$|L(0)| + |DL(0)| \le C_*$$

*and*

$$C_* \le C\left\{|u|_{L^\infty(B_1)} + \sup_{0<r<1} r^{1-\alpha}\left(\frac{1}{|B_r|}\int_{B_r}|f|^n\right)^{\frac{1}{n}}\right\}$$

*where $C$ is a positive constant depending only on $n, \lambda, \Lambda$ and $\alpha$.*

## §4. $W^{2,p}$ Estimates

In this section we will prove the $W^{2,p}$ estimates for viscosity solutions.

Throughout this section we always assume that $a_{ij} \in C(B_1)$ satisfies

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \text{ for any } x \in B_1 \text{ and any } \xi \in \mathbb{R}^n$$

for some positive constants $\lambda$ and $\Lambda$ and that $f$ is a continuous function in $B_1$.

The main result in this section is the following theorem.

**Theorem 4.1.** *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \textit{in } B_1.$$

*Then for any $p \in (n, \infty)$ there exists an $\varepsilon > 0$, depending only on $n, \lambda, \Lambda$ and $p$, such that if*

$$\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|a_{ij} - a_{ij}(x_0)|^n\right)^{\frac{1}{n}} \le \varepsilon \quad \textit{for any } B_r(x_0) \subset B_1,$$

*then $u \in W^{2,p}_{loc}(B_1)$. Moreover there holds*

$$\|u\|_{W^{2,p}(B_{\frac{1}{2}})} \le C\left\{|u|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right\}$$

*where $C$ is a positive constant depending only on $n, \lambda, \Lambda$ and $p$.*

As before we prove the following result instead.

**Lemma 4.2.** *Suppose $u \in C(B_{8\sqrt{n}})$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_{8\sqrt{n}}.$$

*Then for any $p \in (n, \infty)$ there exist positive constants $\varepsilon$ and $C$, depending only on $n, \lambda, \Lambda$ and $p$, such that if*

$$\|u\|_{L^\infty(B_{8\sqrt{n}})} \le 1, \quad \|f\|_{L^p(B_{8\sqrt{n}})} \le \varepsilon$$

*and*

$$\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |a_{ij} - a_{ij}(x_0)|^n\right)^{\frac{1}{n}} \le \varepsilon \quad \text{for any } B_r(x_0) \subset B_{8\sqrt{n}},$$

*then $u \in W^{2,p}(B_1)$ and $\|u\|_{W^{2,p}(B_1)} \le C$.*

Before the proof we first describe the strategy.

Let $\Omega$ be a bounded domain and $u$ be a continuous function in $\Omega$. As in Section 2 we define for $M > 0$

$$\begin{aligned} G_M(u, \Omega) = \{x_0 \in \Omega;\ & \text{there exists an affine function } L \text{ such that} \\ & L(x) - \frac{M}{2}|x - x_0|^2 \le u(x) \le L(x) + \frac{M}{2}|x - x_0|^2 \\ & \text{for } x \in \Omega \text{ with equality at } x_0\} \\ A_M(u, \Omega) = \Omega \setminus & G_M(u, \Omega). \end{aligned}$$

We consider the function

$$\theta(x) = \theta(u, \Omega)(x) = \inf\{M; x \in G_M(u, \Omega)\} \in [0, \infty] \quad \text{for } x \in \Omega.$$

It is straightforward to verify that for $p \in (1, \infty]$ the condition $\theta \in L^p(\Omega)$ implies $D^2u \in L^p(\Omega)$ and

$$\|D^2u\|_{L^p(\Omega)} \le 2\|\theta\|_{L^p(\Omega)}.$$

In order to study the integrability of the function $\theta$ we discuss its distribution function, i.e.,

$$\mu_\theta(t) = |\{x \in \Omega; \theta(x) > t\}| \quad \text{for any } t > 0.$$

It is clear that

$$\mu_\theta(t) \le |A_t(u, \Omega)| \quad \text{for any } t > 0.$$

Hence we need to study the decay of $|A_t(u, \Omega)|$.

**Lemma 4.3.** *Suppose that $\Omega$ is a bounded domain with $B_{8\sqrt{n}} \subset \Omega$ and that $u \in C(\Omega)$ is a viscosity solution of*

$$a_{ij} D_{ij} u = f \quad \text{in } B_{8\sqrt{n}}.$$

*Then for any $\varepsilon_0 \in (0,1)$ there exist an $M > 1$, depending only on $n, \lambda$ and $\Lambda$, and an $\varepsilon \in (0,1)$, depending only on $n, \lambda, \Lambda$ and $\varepsilon_0$, such that if*

$$\|f\|_{L^n(B_{8\sqrt{n}})} \le \varepsilon, \quad \|a_{ij} - a_{ij}(0)\|_{L^n(B_{7\sqrt{n}})} \le \varepsilon \tag{1}$$

*and*

$$G_1(u, \Omega) \cap Q_3 \ne \phi, \tag{2}$$

*then there holds*

$$|G_M(u, \Omega) \cap Q_1| \ge 1 - \varepsilon_0.$$

*Proof.* Let $x_1 \in G_1(u, \Omega) \cap Q_3$. Then there exists an affine function $L$ such that

$$-\frac{1}{2}|x - x_1|^2 \le u(x) - L(x) \le \frac{1}{2}|x - x_1|^2 \text{ in } \Omega.$$

By considering $(u - L)/c(n)$ instead of $u$, for $c(n) > 1$ large enough, depending only on $n$, we may assume that

$$|u| \le 1 \quad \text{in } B_{8\sqrt{n}} \tag{3}$$

which implies

$$-|x|^2 \le u(x) \le |x|^2 \quad \text{for any } x \in \Omega \setminus B_{6\sqrt{n}}. \tag{4}$$

Solve for $h \in C(\bar{B}_{7\sqrt{n}}) \cap C^\infty(B_{7\sqrt{n}})$ such that

$$\begin{aligned} a_{ij}(0) D_{ij} h = 0 &\quad \text{in } B_{7\sqrt{n}} \\ h = u &\quad \text{on } \partial B_{7\sqrt{n}}. \end{aligned}$$

Then Lemma 3.1 implies

$$|u - h|_{L^\infty(B_{6\sqrt{n}})} \le C\Big\{\varepsilon^\gamma + \|f\|_{L^n(B_{8\sqrt{n}})}\Big\} \tag{5}$$

and

$$\|h\|_{C^2((B_{6\sqrt{n}})} \le C \tag{6}$$

where $C > 0$ and $\gamma \in (0,1)$, as in Lemma 3.1, depending only on $n, \lambda$ and $\Lambda$. Consider $h|_{\bar{B}_{6\sqrt{n}}}$. Extend $h$ outside $\bar{B}_{6\sqrt{n}}$ continuously such that $h = u$ in $\Omega \setminus B_{7\sqrt{n}}$ and $|u - h|_{L^\infty(\Omega)} = |u - h|_{L^\infty(B_{6\sqrt{n}})}$. Note $|h| \le 1$ in $\Omega$. It follows that $|u - h|_{L^\infty(\Omega)} \le 2$ and hence with (4)

$$-2 - |x|^2 \le h(x) \le 2 + |x|^2 \quad \text{for any } x \in \Omega \setminus \bar{B}_{6\sqrt{n}}.$$

Then there exists an $N > 1$, depending only on $n, \lambda$ and $\Lambda$, such that

$$Q_1 \subset G_N(h, \Omega). \tag{7}$$

Consider

$$w = \frac{\min\{1, \delta_0\}}{2C\varepsilon^\gamma}(u - h)$$

where $\delta_0$ is the constant in Lemma 2.6 and $C$ and $\gamma$ are constants in (5) and (6). It is easy to check that $w$ satisfies the hypothesis of Lemma 2.6 in $\Omega$. We may apply Lemma 2.6 to get

$$|A_t(w, \Omega) \cap Q_1| \le Ct^{-\mu} \text{ for any } t > 0.$$

Therefore we have

$$|A_s(u - h, \Omega) \cap Q_1| \le C\varepsilon^{\gamma\mu} s^{-\mu} \text{ for any } s > 0.$$

It follows that

$$|G_N(u - h, \Omega) \cap Q_1| \ge 1 - C_1\varepsilon^{\gamma\mu} \ge 1 - \varepsilon_0$$

if we choose $\varepsilon = \varepsilon(n, \lambda, \Lambda, \varepsilon_0) \in (0,1)$ small. With (7) we get

$$|G_{2N}(u, \Omega) \cap Q_1| \ge 1 - \varepsilon_0.$$

*Remark.* In fact we prove the Lemma 4.3 with the assumption (2) replaced by (3).

*Proof of Lemma 4.2.*

Step I. For any $\varepsilon_0 \in (0,1)$ there exist an $M > 1$, depending only on $n, \lambda$ and $\Lambda$, and an $\varepsilon \in (0,1)$, depending only on $n, \lambda, \Lambda$ and $\varepsilon_0$, such that under the assumptions of Lemma 4.2 there holds

$$|G_M(u, B_{8\sqrt{n}}) \cap Q_1| \ge 1 - \varepsilon_0. \tag{1}$$

We remark that $M$ does not depend on $\varepsilon_0$.

In fact we have $|u| \le 1 \le |x|^2$ in $B_{8\sqrt{n}} \setminus B_{6\sqrt{n}}$. We may apply Lemma 4.3 to get (1) with $\Omega = B_{8\sqrt{n}}$ (see the Remark after the Lemma 4.3).

Step II. We set, for $k = 0, 1, \cdots$,

$$\begin{aligned} A &= A_{M^{k+1}}(u, B_{8\sqrt{n}}) \cap Q_1 \\ B &= \big(A_{M^k}(u, B_{8\sqrt{n}}) \cap Q_1\big) \cup \big\{x \in Q_1; m(f^n)(x) \ge (c_1 M^k)^n\big\} \end{aligned}$$

for some $c_1 > 0$ to be determined, depending only on $n, \lambda, \Lambda$ and $\varepsilon_0$. Then there holds

$$|A| \le \varepsilon_0 |B|.$$

The proof is identical to that of Lemma 2.6.

Step III. We finish the proof of Lemma 4.3. We take $\varepsilon_0$ such that

$$\varepsilon_0 M^p = \frac{1}{2}$$

where $M$, depending only on $n, \lambda$ and $\Lambda$, is as in Step I. Hence the constants $\varepsilon$ and $c_1$ depend only on $n, \lambda, \Lambda$ and $p$. Define for $k = 0, 1, \cdots$,

$$\begin{aligned} \alpha_k &= |A_{M^k}(u, B_{8\sqrt{n}}) \cap Q_1| \\ \beta_k &= \big|\big\{x \in Q_1; m(f^n)(x) \ge (c_1 M^k)^n\big\}\big|. \end{aligned}$$

Then Step II implies $\alpha_{k+1} \le \varepsilon_0(\alpha_k + \beta_k)$ for any $k = 0, 1, \cdots$. Hence by iteration we have

$$\alpha_k \le \varepsilon_0^k + \sum_{i=1}^{k-1} \varepsilon_0^{k-i} \beta_i.$$

Since $f^n \in L^{p/n}$ and the maximal operator is of strong type $(p, p)$, we conclude that $m(f^n) \in L^{p/n}$ and

$$\|m(f^n)\|_{L^{\frac{p}{n}}} \le C \|f\|_{L^p}^n \le C.$$

Then the definition of $\beta_k$ implies

$$\sum_{k \ge 0} M^{pk} \beta_k \le C.$$

As before we set

$$\theta(x) = \theta(u, B_{\frac{1}{2}})(x) = \inf\{M; x \in G_M(u, B_{\frac{1}{2}})\} \in [0, \infty] \quad \text{for } x \in B_{\frac{1}{2}}$$

and

$$\mu_\theta(t) = |\{x \in B_{\frac{1}{2}}; \theta(x) > t\}| \quad \text{for any } t > 0.$$

The proof will be finished if we show

$$\|\theta\|_{L^p(B_{\frac{1}{2}})} \le C.$$

It is clear that

$$\mu_\theta(t) \le |A_t(u, B_{\frac{1}{2}})| \le |A_t(u, B_{8\sqrt{n}}) \cap Q_1| \quad \text{for any } t > 0.$$

It suffices to prove, with the definition of $\alpha_k$, that

$$\sum_{k\ge 1} M^{pk}\alpha_k \le C.$$

In fact we have

$$\begin{aligned}\sum_{k\ge 1} M^{pk}\alpha_k &\le \sum_{k\ge 1}(\varepsilon_0 M^p)^k + \sum_{k\ge 1}\sum_{i=0}^{k-1} \varepsilon_0^{k-i} M^{p(k-i)} M^{pi}\beta_i \\ &\le \sum_{k\ge 1} 2^{-k} + \big(\sum_{i\ge 0} M^{pi}\beta_i\big)\big(\sum_{j\ge 1} 2^{-j}\big) \le C.\end{aligned}$$

This finishes the proof.

# CHAPTER 6
# MINIMIZERS

Under appropriate assumptions weak solutions to elliptic equations of divergence forms can be viewed as minimizers of convex functionals. While nondifferential functionals have no Euler-Lagrange equations. In this chapter we will discuss regularity of minimizers for such functionals. First we will prove that minimizers satisfy the Harnack inequality, and hence are Hölder continuous. This generalizes the similar results due to DeGiorgi for weak solutions. Second we will improve the integrability of gradients for minimizers. This kind of results play an important role in the regularity theory of elliptic systems and quasi-convex functionals.

## §1. The Calculus of Variations

In this section we always assume that $\Omega$ is a bounded connected domain in $\mathbb{R}^n$ and that $m > 1$ is a constant. Suppose that $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Caratheodory function, i.e., $F(x,p)$ is measurable in $x$ for all $p \in \mathbb{R}^n$ and continuous in $p$ for almost all $x \in \Omega$. We begin with the model problem of finding a minimizer for the functional

$$E(w) \equiv \int_\Omega F(x, Dw(x))dx$$

among the class

$$\mathcal{K} \equiv \{w \in W^{1,m}(\Omega);\ w = g \text{ on } \partial\Omega\}$$

for some fixed function $g : \partial\Omega \to \mathbb{R}$.

What structural assumption on the nonlinearity $F$ allows the existence of minimizers? For an answer we assume that $u \in \mathcal{K}$ is a minimizer and that $F$ is at least $C^2$, and then set

$$e(t) \equiv E(u + t\varphi) = \int_\Omega F(x, Du + tD\varphi)dx$$

where $\varphi$ is a Lipschitz function with compact support in $\Omega$ and $t \in \mathbb{R}$. Since $e$ attains its minimum at 0, we have

$$e''(0) = \int_\Omega \frac{\partial^2 F}{\partial p_i \partial p_j}(x, Du)D_i\varphi D_j\varphi dx \geq 0.$$

We may set

$$\varphi(x) = \varepsilon\eta(x)\rho\Big(\frac{x\cdot\xi}{\varepsilon}\Big)$$

for $\eta \in C_0^\infty(\Omega)$, $\xi \in \mathbb{R}^n$ and $\rho$ the 2-periodic sawtooth function equaling $x$ in $[0,1]$ and $2-x$ in $[1,2]$. Substituting such $\varphi$ and letting $\varepsilon \to 0$, we obtain

$$\frac{\partial^2 F}{\partial p_i \partial p_j}(x, Du(x))\xi_i\xi_j \geq 0 \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

This inequality strongly suggests that it is natural to assume $F$ is convex in terms of $p$.

*Remark.* Under appropriate growth assumptions on $F$, we have for minimizer $u$

$$0 = e'(0) = \int_\Omega \frac{\partial F}{\partial p_i}(x, Du)D_i\varphi dx \quad \text{for any } \varphi \in C_0^1(\Omega).$$

So our minimizer $u$ is a weak solution of the *Euler − Lagrange equation*

$$\begin{aligned} -\mathrm{div}(D_pF(x,Du)) &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

It is obvious that the convexity of functionals is equivalent to the ellipticity of the corresponding equations. A special case is given if the integrand $F$ is a quadratic form, i.e.,

$$F(x,p) = a_{ij}(x)p_ip_j$$

for some symmetric matrix $\{a_{ij}(x)\}$ defined in $\Omega$. Then the convexity of $F$ is equivalent to the requirement that $\{a_{ij}(x)\}$ is semi-positive definite, that is,

$$a_{ij}(x)\xi_i\xi_j \geq 0 \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

¿From the above remark we may get various regularity results for minimizers of convex functional if $F$ is differentiable and satisfies appropriate growth conditions. The goal of this chapter is to recover part of the results without the assumptions on convexity and differentiability.

We assume the following growth condition and coerciveness condition on $F$: there exist two positive constants $\lambda$ and $\Lambda$ and a nonnegative constant $\chi$ such that

$$(*) \qquad \lambda|p|^m - \chi \leq F(x,p) \leq \Lambda|p|^m + \chi \quad \text{for any } (x,p) \in \Omega \times \mathbb{R}^n$$

where $m > 1$ is a constant.

The function $u \in W^{1,m}(\Omega)$ is a (local) minimizer of the functional

$$E(w) = \int_\Omega F(x, Dw)dx$$

if

$$\int_\Omega F(x,Du)dx \leq \int_\Omega F(x, Du + D\varphi)dx \quad \text{for any } \varphi \in W_0^{1,m}(\Omega).$$

In fact we may integrate over the set $\mathrm{supp}(\varphi)$.

We first derive a Caccioppoli-type inequality.

**Lemma 1.1.** *Suppose that $u \in W^{1,m}(\Omega)$ is a minimizer of the functional $E(u)$ and that $F$ satisfies $(*)$. Then there holds for any $0 < r < R$ and any $k \in \mathbb{R}$ with $B_R \subset \Omega$*

$$\int_{B_r} |Du|^m \leq C\left\{\frac{1}{(R-r)^m}\int_{B_R} |u-k|^m + \chi|B_R|\right\}$$

*where $C$ is a positive constant depending only on $m, n, \lambda$ and $\Lambda$.*

*Proof.* Choose $\eta \in C_0^\infty(B_R)$ with $\eta \equiv 1$ in $B_r$ and $0 \leq \eta \leq 1$ and $|D\eta| \leq 2(R-r)^{-1}$ in $B_R$. Set $\varphi = \eta(u-k)$ for some $k \in \mathbb{R}$. Then by minimality we have

$$\int_{B_R} F(x, Du)dx \leq \int_{B_R} F(x, Du + D\varphi)dx$$

which implies by the assumption $(*)$

$$\int_{B_R} \eta^m|Du|^m \leq C\left\{\int_{B_R} (1-\eta)^m|Du|^m + \int_{B_R} |D\eta|^m|u-k|^m + \chi|B_R|\right\}$$

where $C$ is a positive constant depending only on $m, n, \lambda$ and $\Lambda$. By the choice of $\eta$ we obtain

$$\int_{B_r} |Du|^m \leq C\left\{\int_{B_R\setminus B_r} |Du|^m + \frac{1}{(R-r)^m}\int_{B_R} |u-k|^m + \chi|B_R|\right\}.$$

Now we add $C$ times the left-hand side and get

$$\int_{B_r} |Du|^m \leq \theta\int_{B_R} |Du|^m + C\left\{\frac{1}{(R-r)^m}\int_{B_R} |u-k|^m + \chi|B_R|\right\}$$

where $\theta \in (0,1)$ is a constant depending only on $m, n, \lambda$ and $\Lambda$. By Lemma 1.2 in Chapter 4 we get the result.

*Remark.* In the above proof we may take $\varphi = \eta(u-k)^\pm$. Then we get the following estimates

$$\int_{B_r} |D(u-k)^+|^m \leq C\left\{\frac{1}{(R-r)^m}\int_{B_R} |(u-k)^+|^m + \chi|B_R \cap \{u \geq k\}|\right\}$$

and

$$\int_{B_r} |D(u-k)^-|^m \leq C\left\{\frac{1}{(R-r)^m}\int_{B_R} |(u-k)^-|^m + \chi|B_R \cap \{u \leq k\}|\right\}.$$

In Section 2 we will prove that such $u$ satisfies Harnack inequality and hence is Hölder continuous.

**Corollary 1.2.** *Suppose $u$ is as in Lemma 1.1. Then there holds for any $B_R \subset \Omega$*

$$\left(\frac{1}{|B_{\frac{R}{2}}|}\int_{B_{\frac{R}{2}}} |Du|^m\right)^{\frac{1}{m}} \le C\left\{\left(\frac{1}{|B_R|}\int_{B_R} |Du|^q\right)^{\frac{1}{q}} + \chi\right\}$$

*where $q = \frac{mn}{m+n} < m$ and $C$ is a positive constant depending only on $m, n, \lambda$ and $\Lambda$.*

*Remark.* If $\chi = 0$ this is the reversed Hölder inequality except for the fact that integration is made on different sets. We may rewrite the result as

$$\left(\frac{1}{|B_{\frac{R}{2}}|}\int_{B_{\frac{R}{2}}} (|Du| + \chi)^m\right)^{\frac{1}{m}} \le C\left(\frac{1}{|B_R|}\int_{B_R} (|Du| + \chi)^q\right)^{\frac{1}{q}}.$$

In Section 3 we will prove that $Du \in L^p_{\mathrm{loc}}$ for some $p > m$ depending only on $m, n, \lambda$ and $\Lambda$.

*Proof.* By taking $r = R/2$ and $k = u_R$ in Lemma 1.1 we have

$$\int_{B_{\frac{R}{2}}} |Du|^m \le C\left\{\frac{1}{R^m}\int_{B_R} |u - u_R|^m + \chi|B_R|\right\}$$

where $C$ is a positive constant depending only on $m, n, \lambda$ and $\Lambda$. We may apply the Poincaré inequality to get the result.

## §2. DeGiorgi's Class

In this section we will prove the Harnack inequality for DeGiorgi's class. We always assume that $\Omega$ is a connected domain in $\mathbb{R}^n$ and that $m > 1$ is a constant.

**Definition.** For $u \in W^{1,m}(\Omega)$, define $u \in DG^+(\Omega)$ if for any $0 < r < R$ and any $k \in \mathbb{R}$ with $B_R \subset \Omega$ there holds

$$\int_{B_r} |D(u-k)^+|^m \le c_0\left\{\frac{1}{(R-r)^m}\int_{B_R} |(u-k)^+|^m + \chi^m |B_R \cap \{u \ge k\}|\right\}$$

and $u \in DG^-(\Omega)$ if for any $0 < r < R$ and any $k \in \mathbb{R}$ with $B_R \subset \Omega$ there holds

$$\int_{B_r} |D(u-k)^-|^m \le c_0\left\{\frac{1}{(R-r)^m}\int_{B_R} |(u-k)^-|^m + \chi^m |B_R \cap \{u \le k\}|\right\}$$

where $c_0$ is a positive constant and $\chi$ is a nonnegative constant.

We further define the DeGiorgi class $DG(\Omega) = DG^+(\Omega) \cap DG^-(\Omega)$.

We may define DeGiorgi class in an equivalent way. We set

$$
\begin{aligned}
A(k, r) &= \{x \in B_r;\ u(x) \geq k\} \\
D(k, r) &= \{x \in B_r;\ u(x) \leq k\}.
\end{aligned}
$$

Then $u \in DG^+(\Omega)$ if for any $0 < r < R$ and any $k \in \mathbb{R}$ with $B_R \subset \Omega$ there holds

$$
\int_{A(k,r)} |Du|^m \leq c_0 \left\{ \frac{1}{(R-r)^m} \int_{A(k,R)} |u-k|^m + \chi^m |A(k,R)| \right\}
$$

and $u \in DG^-(\Omega)$ if for any $0 < r < R$ and any $k \in \mathbb{R}$ with $B_R \subset \Omega$ there holds

$$
\int_{D(k,r)} |Du|^m \leq c_0 \left\{ \frac{1}{(R-r)^m} \int_{D(k,R)} |u-k|^m + \chi^m |D(k,R)| \right\}
$$

where $c_0$ is a positive constant.

*Remark.* For $m > n$ Sobolev embedding implies the Hölder continuity of $u$ for $u \in W^{1,m}$. In this section we will concentrate on the case $m < n$. For $m = n$ the proof may be modified.

We first prove the local boundedness for functions in DeGiorgi class.

**Theorem 2.1.** *Suppose $u \in DG^+(\Omega)$ with $B_R \subset \Omega$. Then for any $\sigma \in (0,1)$ and any $p > 0$ there holds*

$$
\sup_{B_{\sigma R}} u^+ \leq \frac{C}{(1-\sigma)^{\frac{n}{p}}} \left\{ \left( \frac{1}{|B_R|} \int_{B_R} (u^+)^p \right)^{\frac{1}{p}} + \chi R \right\}
$$

*where $C$ is a positive constant depending only on $m, n, c_0$ and $p$.*

*Proof.* We normalize so that $R = 1$. We prove for $p = m$ first. For any $0 < r < R < 1$ we set $\bar{r} = (R + r)/2$ and take $\eta \in C_0^\infty(B_{\bar{r}})$ with $\eta \equiv 1$ in $B_r$ and $|D\eta| \leq 4(R-r)^{-1}$. Note $r < \bar{r} < R$. Then with $m^* = mn/(n-m)$ we have by Hölder inequality and Sobolev inequality

$$
\begin{aligned}
\int_{B_r} |(u-k)^+|^m &\leq \int_{B_{\bar{r}}} |(u-k)^+ \eta|^m \\
&\leq \left( \int_{B_{\bar{r}}} |(u-k)^+ \eta|^{m^*} \right)^{\frac{m}{m^*}} |A(k, \bar{r})|^{1-\frac{m}{m^*}} \\
&\leq c(m,n) \left\{ \int_{B_{\bar{r}}} |D(u-k)^+ \eta|^m + \int_{B_{\bar{r}}} |(u-k)^+ D\eta|^m \right\} |A(k, \bar{r})|^{\frac{m}{n}}
\end{aligned}
$$

i.e.,

$$\int_{A(k,r)}(u-k)^m \le C\Big\{\int_{A(k,\bar r)}|Du|^m + \frac{1}{(R-r)^m}\int_{A(k,R)}(u-k)^m\Big\}|A(k,R)|^{\frac{m}{n}}.$$

By definition of $DG^+(\Omega)$ we obtain, with r replaced by $\bar r$, for any $0<r<R\le 1$ and any $k>0$

$$\int_{A(k,r)}(u-k)^m \le C\Big\{\frac{1}{(R-r)^m}\int_{A(k,R)}(u-k)^m + \chi^m|A(k,R)|\Big\}|A(k,R)|^{\frac{m}{n}}.$$

As in the proof of Theorem 1.1 in Chapter 4 we may prove the result for $p=m$. General case may be obtained by an interpolation argument. For details see Section 1 in Chapter 4.

Next result is the so-called weak Harnack inequality.

**Theorem 2.2.** *Suppose $u\in DG^-(\Omega)$ with $u\ge 0$ in $B_R\subset\Omega$. Then there exists a constant $p=p(m,n,c_0)>0$ such that for any $\sigma,\tau\in(0,1)$ there holds*

$$\left(\frac{1}{|B_{\sigma R}|}\int_{B_{\sigma R}}u^p\right)^{\frac{1}{p}} \le C\Big\{\inf_{B_{\tau R}}u + \chi R\Big\}$$

*where $C$ is a positive constant depending only on $m,n,c_0,\sigma$ and $\tau$.*

Now the Harnack inequality is an easy consequence of Theorem 2.1 and Theorem 2.2.

**Corollary 2.3.** *Suppose $u\in DG(\Omega)$ with $u\ge 0$ in $B_R\subset\Omega$. Then for any $\sigma\in(0,1)$ there holds*

$$\sup_{B_{\sigma R}}u \le C\Big\{\inf_{B_{\sigma R}}u + \chi R\Big\}$$

*where $C$ is a positive constant depending only on $m,n,c_0$ and $\sigma$.*

In the rest of this section we will prove Theorem 2.2. First we need a simple lemma. Recall the notation

$$D(k,r)=\{x\in B_r; u(x)\le k\}.$$

**Lemma 2.4.** *Suppose $u\in W^{1,1}(B_R)$ satisfies*

$$|\{x\in B_R; u(x)\ge k_0\}|\ge \varepsilon|B_R|$$

*for some $\varepsilon\in(0,1)$ and some $k_0\in\mathbb{R}$. Then for all $k<h\le k_0$ there holds*

$$(h-k)|D(k,R)|^{\frac{n-1}{n}} \le c(n,\varepsilon)\int_{D(h,R)\setminus D(k,R)}|Du|$$

*where $c(n,\varepsilon)$ is a positive constant depending only on $n$ and $\varepsilon$.*

*Proof.* For any $k < h \le k_0$ we set $v = \max(u,h) - \max(u,k)$ in $B_R$. Then $\{v = 0\} = \{u \ge h\} \supset \{u \ge k_0\}$. This implies $|B_R \cap \{v = 0\}| \ge \varepsilon |B_R|$. By Sobolev-Poincaré inequality we obtain

$$\Big(\int_{B_R} v^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \le c(n,\varepsilon)\int_{B_R} |Dv|.$$

This finishes the proof.

Now we prove the following density result.

**Lemma 2.5.** *Suppose $u \in DG^-(B_2)$ with $u \ge 0$ in $B_2$. Then for any $\varepsilon \in (0,1)$ there exist constants $M > 1$ and $\chi_0 > 0$, depending only on $m, n, c_0$ and $\varepsilon$, such that if $\chi \le \chi_0$ and*

$$|\{x \in B_2;\ u(x) \ge M\}| \ge \varepsilon |B_2|$$

*then there holds*

$$\inf_{B_1} u \ge 1.$$

*Proof.* The proof consists of two steps.

Step I. For any $\varepsilon, \delta \in (0,1)$ there exist constants $\chi_0 > 0$ and $M \ge 2$, depending only on $m, n, c_0, \delta$ and $\varepsilon$, such that if $\chi \le \chi_0$ and

$$|\{x \in B_2;\ u(x) \ge M\}| \ge \varepsilon |B_2|$$

then there holds

$$|\{x \in B_1;\ u(x) \ge 2\}| \ge \delta |B_1|.$$

In the definition of $DG^-(\Omega)$ we take $r = 1$ and $R = 2$. Hence for any $k \ge 0$ there holds

$$\int_{D(k,1)} |Du|^m \le C\{k^m + \chi^m\}|D(k,2)| \le C(k+\chi)^m.$$

Lemma 2.4 implies for any $0 < k < h \le M$

$$\begin{aligned}(h-k)|D(k,1)|^{1-\frac{1}{n}} &\le C(\varepsilon)\int_{D(h,1)\setminus D(k,1)} |Du| \\ &\le C(\varepsilon)\Big(\int_{D(h,1)} |Du|^m\Big)^{\frac{1}{m}} |D(h,1)\setminus D(k,1)|^{1-\frac{1}{m}} \\ &\le C(\varepsilon)(h+\chi)|D(h,1)\setminus D(k,1)|^{1-\frac{1}{m}}\end{aligned}$$

in particular

$$(h-k)|D(k,1)| \leq C(h+\chi)|D(h,1)\setminus D(k,1)|^{1-\frac{1}{m}}$$

Therefore we obtain for any $0 < k < h \leq M$

$$|D(k,1)|^{\frac{m}{m-1}} \leq C\left(\frac{h+\chi}{h-k}\right)^{\frac{m}{m-1}}|D(h,1)\setminus D(k,1)|.$$

We may assume that $M = 2^N$ for some positive integer $N$ to be determined. Choose the iteration as

$$h = \frac{M}{2^{\ell-1}},\ k = \frac{M}{2^{\ell}} \quad \text{for } \ell = 1, 2, \cdots, N-1.$$

This implies for $\ell = 1, 2, \cdots, N-1$,

$$\begin{aligned}|D(\frac{M}{2^{\ell}},1)|^{\frac{m}{m-1}} &\leq C\big(2+\frac{2^{\ell}\chi}{M}\big)^{\frac{m}{m-1}}|D(\frac{M}{2^{\ell-1}},1)\setminus D(\frac{M}{2^{\ell}},1)| \\ &\leq C(2+\chi)^{\frac{m}{m-1}}|D(\frac{M}{2^{\ell-1}},1)\setminus D(\frac{M}{2^{\ell}},1)|.\end{aligned}$$

Note $D(\frac{M}{2^{\ell}},1) \subset D(\frac{M}{2^{\ell-1}},1)$. We add $\ell$ from 1 to $N-1$ and get

$$(N-1)|D(\frac{M}{2^{N-1}},1)|^{\frac{m}{m-1}} \leq C(2+\chi)^{\frac{m}{m-1}}|D(M,2)| \leq C(\varepsilon)(1-\varepsilon)(2+\chi)^{\frac{m}{m-1}}.$$

Therefore we obtain

$$|D(2,1)| \leq \frac{C(\varepsilon)(2+\chi)^{\frac{m}{m-1}}}{N-1}|B_1|.$$

We may choose $N$ large such that

$$\frac{C(\varepsilon)(2+\chi_0)^{\frac{m}{m-1}}}{N-1} \leq 1-\delta.$$

Spet II. There exist constants $\delta_0 \in (1/2, 1)$ and $\chi_0 > 0$, depending only on $m, n$ and $c_0$, such that if $\chi \leq \chi_0$ and

$$|\{x \in B_1;\ u(x) \geq 2\}| \geq \delta_0|B_1|$$

then there holds

$$\inf_{B_1} u \geq 1.$$

First we have, by assumption, for any $R \in [1, 2]$

$$|\{x \in B_R;\ u(x) \geq 2\}| \geq \frac{\delta_0}{2^n}|B_R| \geq \frac{1}{2^{n+1}}|B_R|.$$

By definition of $DG^-(\Omega)$ there holds for any $1 \le r < R \le 2$ and $1 \le k < h \le 2$

$$\int_{D(h,r)} |Du|^m \le c_0 \left\{ \frac{1}{(R-r)^m} h^m + \chi^m \right\} |D(h,R)| \le C \left( \frac{h+\chi}{R-r} \right)^m |D(h,R)|.$$

Lemma 2.4 implies

$$\begin{aligned}(h-k)|D(k,r)|^{\frac{n-1}{n}} &\le C \int_{D(h,r)\setminus D(k,r)} |Du| \\ &\le C \left( \int_{D(h,r)} |Du|^m \right)^{\frac{1}{m}} |D(h,r)|^{1-\frac{1}{m}}.\end{aligned}$$

Hence we obtain

$$(h-k)|D(k,r)|^{1-\frac{1}{n}} \le \frac{C}{R-r}(h+\chi)|D(h,R)|$$

or

$$|D(k,r)|^{1-\frac{1}{n}} \le \frac{C}{R-r} \cdot \frac{h+\chi}{h-k} |D(h,R)|.$$

With $\gamma = (n-1)^{-1}$ we obtain for any $1 \le r < R \le 2$ and $1 \le k < h \le 2$

$$|D(k,r)| \le \frac{C}{(R-r)^{1+\gamma}} \left( \frac{h+\chi}{h-k} \right)^{1+\gamma} |D(h,R)|^{1+\gamma}.$$

As in Chapter 4 we set for $\ell = 0, 1, 2, \cdots$,

$$r_\ell = 1 + \frac{1}{2^\ell} \quad \text{and} \quad k_\ell = 1 + \frac{1}{2^\ell}.$$

Then we may prove by induction that

$$|D(k_\ell, r_\ell)| \le \frac{1}{a^\ell} |D(k_0, r_0)| \quad \text{for any } \ell = 0, 1, 2, \cdots, \tag{1}$$

for some constant $a > 1$, depending only on $n$, provided $|D(k_0, r_0)| = |D(2,2)|$ is small, specifically,

$$C(2+\chi_0)^{1+\gamma}|D(2,2)|^\gamma \le 1.$$

Hence we may choose $\delta_0 > 0$ close to 1 such that

$$C(2+\chi_0)^{1+\gamma}(1-\delta_0)^\gamma \le 1.$$

Letting $\ell \to \infty$ in (1) we conclude that $|D(1,1)| = 0$ or $u \ge 1$ in $B_1$.

In the following we use cubes instead of balls. We rewrite Lemma 2.5 in the following form for convenience.

**Corollary 2.6.** *Suppose $u \in DG^-(B_{3\sqrt{n}})$ with $u \geq 0$ in $B_{3\sqrt{n}}$. Then there exist constants $\chi_0 > 0$, $\varepsilon \in (0,1)$ and $M \geq 1$, depending only on $m, n$ and $c_0$, such that if $\chi \leq \chi_0$ and $\inf_{Q_3} u \leq 1$ there holds*

$$|\{x \in Q_1;\ u(x) \leq M\}| > \varepsilon.$$

*Proof.* Note $Q_3 \subset B_{3\sqrt{n}/2} \subset B_{3\sqrt{n}}$ and the following implication for any $\mu \in (0,1)$

$$|\{x \in Q_1;\ u \geq M\}| \geq \mu |Q_1| = \frac{\mu}{c(n)} |B_{3\sqrt{n}}|$$
$$\Longrightarrow |\{x \in B_{3\sqrt{n}},\ u \geq M\}| \geq \frac{\mu}{c(n)} |B_{3\sqrt{n}}|.$$

We apply Lemma 2.5 to $u$ in $B_{3\sqrt{n}}$ by contradiction argument.

Now we proceed exactly as in Section 2 in Chapter 5 and we get the power decay of the distribution function.

**Lemma 2.7.** *Suppose $u \in DG^-(B_{3\sqrt{n}})$ with $u \geq 0$ in $B_{3\sqrt{n}}$. Then there exist positive constants $\gamma$ and $C$, depending only on $m, n$ and $c_0$, such that*

$$|\{x \in Q_1;\ u(x) > t\}| \leq C t^{-\gamma} \big( \inf_{Q_{\frac{1}{2}}} u + \chi \big)^{\gamma} \quad \textit{for any } t > 0.$$

*Proof.* We will prove that there exist positive constants $\chi_0, \gamma$ and $C$, depending only on $m, n$ and $c_0$, such that if $\chi \leq \chi_0$ and $\inf_{Q_{1/2}} u \leq 1$ there holds

$$|\{x \in Q_1;\ u(x) > t\}| \leq C t^{-\gamma} \quad \text{for any } t > 0.$$

We omit the proof. For details see Section 2 in Chapter 5.

**Corollary 2.8.** *Suppose $u \in DG^-(B_{3\sqrt{n}})$ with $u \geq 0$ in $B_{3\sqrt{n}}$. Then there exist positive constants $p$ and $C$, depending only on $m, n$ and $c_0$, such that*

$$\left( \int_{Q_1} u^p \right)^{\frac{1}{p}} \leq C \left\{ \inf_{Q_{\frac{1}{2}}} u + \chi \right\}.$$

*Proof.* Set $A(t) = \{x \in Q_1;\ u(x) > t\}$ for any $t > 0$. Then we have by Lemma 2.7 for any $\xi > 0$

$$\begin{aligned}
\int_{Q_1} u^p &= p \int_{\xi}^{\infty} t^{p-1} |A(t)| dt + \xi^p |A(\xi)| \\
&\leq C \left\{ p \int_{\xi}^{\infty} t^{p-\gamma-1} dt + \xi^{p-\gamma} \right\} \left( \inf_{B_{\frac{1}{2}}} u + \chi \right)^{\gamma} \\
&= C \xi^{p-\gamma} \left( \inf_{B_{\frac{1}{2}}} u + \chi \right)^{\gamma}
\end{aligned}$$

if we choose $p < \gamma$. Next we may choose

$$\xi = \inf_{Q_{\frac{1}{2}}} u + \chi.$$

This finishes the proof.

## §3. Reversed Hölder Inequality

In this section we will prove that a function $u$ is $L^p$-integrable for some $p > q$ if the $L^q$-average over cubes do not exceed the $L^1$-average over suitable cubes. Recall $Q_r(x_0)$ denotes the cube centered at $x_0$ with side length $r$.

**Theorem 3.1.** *Suppose $u \in L^q(Q)$, for some cube $Q \subset \mathbb{R}^n$ and some $q > 1$, satisfies*

$$\left( ⨍_{Q_r(x_0)} u^q \right)^{\frac{1}{q}} \le c_0 ⨍_{Q_{2r}(x_0)} u \quad \textit{for any } Q_{2r}(x_0) \subset Q$$

*for some $c_0 > 1$. Then $u \in L^p_{\text{loc}}(Q)$ for some $p = p(q, n, c_0) > q$ and there holds*

$$\left( ⨍_{Q_r(x_0)} u^p \right)^{\frac{1}{p}} \le C \left( ⨍_{Q_{2r}(x_0)} u^q \right)^{\frac{1}{q}} \quad \textit{for any } Q_{2r}(x_0) \subset Q$$

*where $C$ is a positive constant depending only on $n, q, c_0$ and dist$(Q_{2r}(x_0), \partial Q)$.*

*Proof.* We normalize so that $⨍_Q u^q = 1$. We assume the cube $Q$ is given by

$$Q = \{x \in \mathbb{R}^n;\ |x_i| < \frac{3}{2},\ i = 1, \cdots, n\},$$

and set

$$C_0 = \{x \in Q;\ |x_i| < \frac{1}{2},\ i = 1, \cdots, n\}$$
$$C_k = \{x \in Q;\ 2^{-k} < \text{dist}(x, \partial Q) \le 2^{-k+1}\}, \quad \text{for } k = 1, 2, \cdots .$$

Obviously we have

$$Q = \cup_{k=0}^{\infty} C_k.$$

Now define

$$\tilde{u}(x) = \frac{1}{(2^k 3)^{\frac{n}{q}}} u(x) \quad \text{for any } x \in C_k$$

and

$$A(t) = \{x \in Q;\ \tilde{u}(x) > t\}.$$

As the first step we prove for any $t \geq 1$

$$\int_{A(t)} \tilde{u}^q \leq C t^{q-1} \int_{A(t)} \tilde{u} \tag{1}$$

where $C$ is a constant depending only on $n, q$ and $c_0$.

Fix $t \geq 1$ and consider $s > t$ to be determined. We have

$$\int_{A(t)} \tilde{u}^q = \int_{A(s)} \tilde{u}^q + \int_{A(t)\setminus A(s)} \tilde{u}^q \leq \int_{A(s)} \tilde{u}^q + \left(\frac{s}{t}\right)^{q-1} t^{q-1} \int_{A(t)} \tilde{u}. \tag{2}$$

Hence we need to estimate $\int_{A(s)} \tilde{u}^q$ in terms of $\int_{A(t)} \tilde{u}$ and to choose $s$ proportionally to $t$.

We begin with a modified Calderon-Zygmund decomposition. We make the first subdivision of $Q$ into $3^n$ unit cubes. Then decompose each unit cube into dyadic cubes. Pick any dyadic cube $P_k^j$ such that (i) $P_k^j$ has length $2^{-k}$ and (ii) $P_k^j \subset C_k$. Note that there are $3^n 2^{nk}$ such cubes in $Q$ and that for each such $P_k^j$ we have

$$\tilde{P}_k^j \subset C_{k-1} \cup C_k \cup C_{k+1} \tag{3}$$

where $\tilde{P}_k^j$ denotes the cube with the same center as $P_k^j$ and the size as twice as that of $P_k^j$. For $s > t \geq 1$ above we have $⨍_Q u^q < s^q$ by the normalization. This implies for each $P_k^j$ above

$$⨍_{P_k} \tilde{u}^q < s^q.$$

We may apply the Calderon-Zygmund decomposition in each $P_k$ to $\tilde{u}^q$ and $s^q$. Hence there exists a sequence of dyadic cubes $\{Q_k^j\}$ such that for any $k, j = 0, 1, \cdots$,

$$\begin{aligned}
&Q_k^j \subset C_k \\
&s^q < ⨍_{Q_k^j} \tilde{u}^q \leq 2^n s^q \\
&\tilde{u} \leq s \quad \text{a.e. in } C_k \setminus \cup_j Q_k^j.
\end{aligned}$$

This implies

$$|A(s) \setminus \cup_{k,j} Q_k^j| = 0$$

and in particular

$$\int_{A(s)} \tilde{u}^q \le \sum_{k,j} \int_{Q_k^j} \tilde{u}^q \le 2^n s^q \sum_{k,j} |Q_k^j| = s^q |\tilde{Q}_k^j|. \tag{4}$$

By assumption, definition of $\tilde{u}$ and (3) we have

$$\begin{aligned} s^q &< \fint_{Q_k^j} \tilde{u}^q = \frac{1}{2^{nk}3^n} \fint_{Q_k^j} u^q \\ &\le \frac{c_0^q}{2^{nk}3^n} \Big( \fint_{\tilde{Q}_k^j} u \Big)^q \le 2^n c_0^q \Big( \fint_{\tilde{Q}_k^j} \tilde{u} \Big)^q . \end{aligned}$$

Therefore we obtain

$$|\tilde{Q}_k^j| \le \frac{2^{\frac{n}{q}} c_0}{s} \int_{\tilde{Q}_k^j} \tilde{u} \le \frac{2^{\frac{n}{q}} c_0}{s} \left\{ \int_{\tilde{Q}_k^j \cap A(t)} \tilde{u} + t|\tilde{Q}_k^j| \right\}.$$

Now choose $s$ such that $2^{n/q} c_0 t = c_* s$ for some $c_* < 1$. This implies that

$$|\tilde{Q}_k^j| \le \frac{c_*}{1-c_*} \cdot \frac{1}{t} \int_{\tilde{Q}_k^j \cap A(t)} \tilde{u}.$$

By Vitali covering lemma there exists a subsequence $\{\tilde{Q}_k^{j'}\}$ of $\{\tilde{Q}_k^j\}$ such that
(i) $\{\tilde{Q}_k^{j'}\}$ has no intersections;
(ii) $|\cup_{k,j} \tilde{Q}_k^j| \le c(n) \sum_{k,j'} |\tilde{Q}_k^{j'}|$.
Therefore we get

$$\begin{aligned} |\cup_{k,j} \tilde{Q}_k^j| \le c(n) \sum_{k,j'} |\tilde{Q}_k^{j'}| &\le c(n) \frac{c_*}{1-c_*} \cdot \frac{1}{t} \sum_{k,j'} \int_{\tilde{Q}_k^{j'} \cap A(t)} \tilde{u} \\ &\le c(n) \frac{c_*}{1-c_*} \cdot \frac{1}{t} \int_{A(t)} \tilde{u}. \end{aligned}$$

With (4) we obtain

$$\int_{A(s)} \tilde{u}^q \le c(n) \frac{c_*}{1-c_*} \cdot \frac{s^q}{t} \int_{A(t)} \tilde{u}.$$

This implies with (2)

$$\int_{A(t)} \tilde{u}^q \le \left\{ c(n) \frac{c_*}{1-c_*} \left(\frac{s}{t}\right)^q + \left(\frac{s}{t}\right)^{q-1} \right\} t^{q-1} \int_{A(t)} \tilde{u}$$

which gives (1).

To continue we set

$$h(t) = \int_{A(t)} \tilde{u} \quad \text{for any } t \geq 1.$$

It is easy to check that for any $r > 1$

$$\int_{A(t)} \tilde{u}^r = -\int_t^\infty s^{r-1} dh(s) \text{ for any } t \geq 1.$$

We may apply Lemma 3.2 in the following, with $q$ replaced by $q-1$, to conclude that for some $p > q$

$$\int_{A(1)} \tilde{u}^p \leq C_* \int_{A(1)} \tilde{u}^q$$

where $C_*$ is a positive constant depending only on $p, q$ and $C$ in (1). Hence we have

$$\int_Q \tilde{u}^p \leq C_* \int_Q \tilde{u}^q.$$

With the normalization assumption this finishes the proof.

We need the following result for Stiltjes integral.

**Lemma 3.2.** *Suppose $h$ is a nonnegative and nonincreasing function in $[1, \infty)$ and that $h$ satisfies for some constants $q, c > 1$*

(1) $$\lim_{t \to \infty} h(t) = 0$$

*and*

(2) $$-\int_t^\infty s^q dh(s) \leq ct^q h(t) \quad \textit{for any } t \geq 1.$$

*Then there holds for any $p \in [q, cq/(c-1))$*

$$-\int_1^\infty s^p dh(s) \leq \frac{q}{cq-(c-1)p}\left(-\int_1^\infty s^q dh(s)\right).$$

*Proof.* First assume that there exists a $k > 1$ such that $h(t) = 0$ for any $t \geq k$. We set

$$I(r) = -\int_1^\infty s^r dh(s) = -\int_1^k s^r dh(s).$$

For any $p > 0$ we obtain by integrating by parts

$$I(p) = I(q) + (p-q)J$$

where by the assumption (2)

$$J = \int_1^k t^{p-q-1}\Big(-\int_t^k s^q dh(s)\Big)dt \le c\int_1^k t^{p-1}h(t)dt \le -\frac{1}{p}I(q) + \frac{c}{p}I(p).$$

Hence we have

$$\Big(p - c(p-q)\Big)I(p) \le qI(q).$$

For the general case the assumption (1) implies for any $k > 1$

$$k^q h(k) \le -\int_k^\infty s^q dh(s).$$

For each $k > 1$ we set

$$h_k(t) = \begin{cases} h(t) & \text{for } t \in [1, k) \\ 0 & \text{for } t \in [k, \infty). \end{cases}$$

Then $h_k$ satisfies the assumptions of Lemma 3.2. By what we just proved we obtain

$$\begin{aligned} -\int_1^k s^p dh(s) \le -\int_1^k s^p dh_k(s) &\le \frac{q}{cq-(c-1)q}\Big(-\int_1^k s^q dh_k(s)\Big) \\ &\le \frac{q}{cq-(c-1)q}\Big(-\int_1^\infty s^q dh(s)\Big). \end{aligned}$$

We may let $k \to \infty$ to get the result.